The Alfven soliton

A. B. Mikhaĭlovskii, V. I. Petviashvili, and A. M. Fridman

I. V. Kurchatov Institute of Atomic Energy
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The possibility of production of an Alfven soliton in a plasma is demonstrated.

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Contemporary plasma theory predicts the existence of magnetosonic[1], Langmuir[2], ion-sound[3], and cyclotron[4] solitons associated with the corresponding plasma oscillation modes. It is also of interest to determine whether solitions associated with other oscillation modes can exist in a plasma. In particular, in view of the important role played by Alfven waves in a plasma, it is of interest to assess the feasibility of an Alfven soliton. This is the subject of the present article.

Just as in the problem of the Langmuir soliton[2] it is necessary to take the finite Debye radius into account, in our problem it is necessary to allow for the finite Larmor radius (FLR) of the ions. The dispersion equation for the Alfven waves with allowance for the FLR of the ions takes, according to [5], the form

$$\epsilon_{\parallel}[1-(\omega/k_z c)^2\epsilon_{\perp}]+(k/k_z)^2\epsilon_{\perp}=0, \quad (1)$$

where

$$\epsilon_{\parallel}=(k_z d)^{-2}, \qquad \epsilon_{\perp}=[1-I_0(Z_i)\exp(-Z_i)](kd)^{-2}. \quad (2)$$

It is assumed that the spacetime dependence of the perturbed quantities is of the form $f_1(x,y,z)=f_1(x)\exp(-i\omega t+ik_x x+ik_y x+ik_y y+ik_z z)$; $k_\perp\equiv(k_x^2+k_y^2)^{1/2}\gg k_z$, $d=(T/4\pi ne^2)^{1/2}$ is the Debye radius; T and n are the plasma temperature and density; $Z_i=k_\perp^2\rho_i^2$, $\rho_i^2=T/m_i\omega_{Bi}^2$ is the square of the Larmor radius of the ions, $\omega_{Bi}=eB/m_i c$ is their cyclotron frequency; B is the stationary magnetic field and is oriented along the z axis; e and m_i are the charge and mass of the ions, c is the speed of light in vacuum; k_x, k_y, and k_z are the components of the wave vector **k**; ω is the oscillation frequency; I_0 is a Bessel function of imaginary argument.

At $Z\ll 1$ we get from (1) and (2)

$$\omega^2=\Omega_A^2\left(1+\frac{7}{4}Z_i\right), \quad (3)$$

where $\Omega_A^2=k_z^2C_A^2$; $C_A^2=B^2/4\pi nm_i$ is the square of the Alfven velocity.

Comparing (3) with the dispersion law of the Langmuir oscillations at $k_z=0$

$$\omega^2=\Omega_L^2\left(1+\frac{3}{2}k_\perp^2 d^2\right), \quad (4)$$

where $\Omega_L^2 = (4\pi e^2 n/m_e)$ is the square of the Langmuir frequency and m_e is the electron mass, we conclude that the existence of an Alfven soliton must be verified in analogy with the problem of the Langmuir soliton. In other words, whereas in the case of the Langmuir soliton we calculate the increment added to Ω_L^2 by the change of the plasma density n as a result of effects quadratic in the wave amplitude, in the case of the Alfven soliton we must calculate the corresponding increment to Ω_A^2.

Putting $B = B_0 + \delta B_z$ and $n = n_0 + \delta n$, we obtain (δn and δB_z are quantities of second order of smallness relative to $|f_1/f_0|$ and depend little on the coordinates and on the time)

$$\Omega_A^2 = \Omega_{A0}^2 + \delta\Omega_A^2 \, , \tag{5}$$

where $\Omega_{A0}^2 = k_z^2 C_{A0}^2$, $C_{A0}^2 = B_0^2/4\pi n_0 m_i$ is the square of the Alfven velocity in the zeroth approximation in the wave amplitude, and

$$\delta\Omega_A^2 = \Omega_{A0}^2\left(\frac{2\delta B_z}{B_0} - \frac{\delta n}{n_0}\right). \tag{6}$$

Using the freezing-in condition $n/B = \text{const}$, we express δn in terms of δB_z

$$\delta n = (n_0/B_0)\delta B_z \, , \tag{7}$$

so that when (5) and (6) are taken into account, Eq. (3) reduces to

$$\omega^2 = \Omega_{A0}^2\left(1 + \frac{7}{4} Z_i + \frac{\delta B_z}{B_0}\right). \tag{8}$$

We obtain δB_z from the pressure-balance equation averaged over the time

$$\delta P + \frac{\overline{B_\perp^2}}{8\pi} + \frac{B_0\,\delta B_z}{4\pi} = 0, \tag{9}$$

where δP is the perturbed pressure, $\mathbf{B}_\perp$ is the perturbation (in first-order perturbation theory) of the transverse magnetic field ($\mathbf{B}_\perp \perp z$), and the bar denotes averaging over the time. We assume that $\delta P = 2\gamma T \delta n$, where $\gamma = 5/3$ for a collision-dominated plasma and $\gamma = 2$ for a collisionless plasma. It follows then from (9), when (7) is taken into account, that

$$\frac{\delta B_z}{B_0} = -\frac{\overline{B_\perp^2}}{(2+\gamma\beta)B_0^2} \, , \tag{10}$$

where $\beta = 16\pi n_0 T/B_0^2$ is the ratio of the plasma pressure to the magnetic-field pressure.

Substituting (10) in (8) we obtain

$$\omega^2 = \Omega_{A0}^2\left[1 + \frac{7}{4} Z_i - \frac{\overline{B_\perp^2}}{(2+\gamma\beta)B_0^2}\right]. \tag{11}$$

The algebraic equation (11) corresponds to the differential equation

$$\omega^2 B_\perp = \Omega_{Ao}^2 \left[1 + \frac{7}{4} \rho_i^2 \left(k_y^2 - \frac{\partial^2}{\partial x^2} \right) - \frac{\widetilde{B_\perp^2}}{(2 + \gamma\beta) B_o^2} \right] B_\perp . \tag{12}$$

We take the x-th component of this equation, using the relation $B_y = (i/k_y)\, \partial B_x / \partial x$, which follows from the equation div**B**$= 0$, and we put

$$B_x / \nu B_o \equiv f, \quad \nu^2 = 7/_4 (2 + \gamma\beta) A^2, \; A^2 = \frac{4}{7}\left(1 + \frac{7}{4} k_y^2 \rho_i^2 - \frac{\omega^2}{k_z^2 C_A^2}\right).$$

We then have in lieu of (12)

$$\frac{\partial^2 f}{\partial \xi^2} = f - \left[f^2 + a^2 \left(\frac{\partial f}{\partial \xi} \right)^2 \right] f , \tag{13}$$

where $\xi = Ax/\rho_i$.

Integrating (13), we obtain

$$a^2 \left(\frac{\partial f}{\partial \xi} \right)^2 = \left(1 + \frac{1}{a^2} \right) \left(1 - e^{-a^2 f^2} \right) - f^2 . \tag{14}$$

In particular, at $a \ll 1$, the solution (14) takes the hyperbolic-secant form

$$f \approx \sqrt{2}\, \mathrm{sch}\, \xi \tag{15}$$

and corresponds to an Alfven perturbation localized in x, in analogy with the cases of the magnetosonic and Langmuir solitons.[1,2] This proves the feasibility of the existence of an Alfven soliton in a plasma.

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