Calculation of the Gell-Mann–Low function in scalar theories with strong nonlinearity

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The Gell-Mann–Low function is obtained in all orders of perturbation theory for a scalar theory with interaction $H_{\text{int}} = g\int(\phi^n/n!)d^Dx$ as $n\to\infty$, $D=2n/(n-2)\to 2$. Its ultraviolet stability points are investigated.

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1. It is known that in quantum electrodynamics and in most hitherto-known quantum field theory models one encounters the physical phenomenon of screening of the interaction due to the polarization of the vacuum, which leads to vanishing of the physical charges, provided that the nonrenormalized charges are small enough.[1] At the same time, in certain presently popular nonAbelian gauge models the situation is the opposite—the nonrenormalized charge vanishes at sufficiently small physical charges.[3] These results were obtained by perturbation theory and therefore have a limited region of applicability. For a final answer to the question of the internal contradiction of quantum electrodynamics and other traditional models of quantum field theory it is necessary to calculate, in all orders of perturbation theory, the so-called Gell-Mann—Low (GML) function,[3] defined as

$$\psi\left(g\left(\frac{k^2}{\mu^2},g_\mu\right)\right)\equiv\frac{dg\left(\frac{k^2}{\mu^2},g_\mu\right)}{d\ln\frac{k^2}{\mu^2}}\tag{1}$$

where $g(k^2/\mu^2, g_\mu)$ is the invariant charge, which is expressed in terms of the renormalized Green's functions and the vertex part [see (3)], μ is the momentum at the normalization point, usually chosen much larger than the masses of all the particles in the theory, and g_μ is the value of the invariant charge at the normalization point $p^2=\mu^2$. The renormalizability of the theory guarantees that the right-side of (1) is independent of $\ln(k^2/\mu^2)$ provided that it is expressed as a function of the invariant charge.

2. We consider in this paper the problem of calculating the GML function in the class of scalar models of quantum field theory described by the Lagrangian

$$L=\int d^Dx\left[\frac{(\partial_\mu\phi)^2}{2}-g\frac{\phi^n}{n!}\right].\tag{2}$$

In order for the theory to be renormalizable, we assume the dimensionality of space D to be equal to

$$D=\frac{2n}{n-2},\qquad n=4,6,8\ldots\ .\tag{2a}$$

This class of models includes the known theories $H_{\rm int} = g\int(\phi^4/4!)\,d^4x$ [4] and $H_{\rm int} = g\int(\phi^6/6!)\,d^3x$.[5]

In the theory with (2), the invariant charge is defined by the formula

$$g\left(\frac{p^2}{\mu^2}, g_\mu\right) = g_\mu \Gamma_c\left(\frac{p^2}{\mu^2}, g_\mu\right) d_c^{n/2}\left(\frac{p^2}{\mu^2}, g_\mu\right); \qquad \Gamma_c(1, g_\mu) = d_c(1, g_\mu) = 1 \tag{3}$$

where Γ_c is the vertex part for the n-point amplitude, d_c/p^2 is the Green's function of the scalar particle. The external invariants for the vertex function Γ_c are chosen in the Euclidean region $p_i^2 = p^2 > 0$, $p_i p_j\,|_{i\neq j} = -p^2/(n-1)$. In this case it is possible to make the Wick rotation $t \to -ix_D$ in the Feynman diagrams. We shall therefore write down from now on all the expressions in Euclidean form.

3. In the theory with $H_{\rm int} = g\int(\phi^4/4!)\,dx^4$, only the first three terms of the expansion of the GML function in the invariant charge are known.[4] In this paper we obtain the GML function in all orders of perturbation theory in the limit of strong nonlinearity of the interaction:

$$n \to \infty, \qquad D = \frac{2n}{n-2} \to 2. \tag{4}$$

One can hope that even for the theory with $H_{\rm int} = g\int(\phi^4/4!)\,d^4x$, and all the more for the theory with $H_{\rm int} = g\int(\phi^6/6!)\,d^3x$, the approximation of the true GML function, given by the limiting transition (4), will be good.

The principal simplification that arises when determining the GML function in the theory (4) is the following: The principal role in the calculation of the invariant charge (3) in each order k of perturbation theory is played by the vertex parts. The sum over the vertex parts with different numbers of external lines n_i in each vertex, and with different internal lines n_{ij} that join the interaction vertices, has at $\tilde{n}_i = \tilde{n}_{ij} = n/k$ a saddle that is broad enough to be able to replace the summation by integration, and narrow enough to be able to take the Feynman diagram outside the summation sign at the saddle point. The contribution of this Feynman diagram contains only a singly-logarithmic ultraviolet divergence, which relieves us of the need of calculating the lower-order logarithm in diagrams containing internal divergences. The calculation of the saddle point with respect to n_i and n_{ij} yields

$$\psi(g) = \sum_{k=2}^{\infty} (-g)^k C_k(n),$$

$$C_k(n)\Big|_{n\to\infty} = \frac{1}{\Gamma(k+1)}\left(\frac{e}{8\pi}\,k^{\frac{k+1}{k-1}}\right)^{n\frac{k-1}{2}} \times \left(2\pi n\frac{k-1}{k}\right)^{-\frac{k}{2}} e^{-(k-1)(1+\ln\pi + c_E)}\sqrt{2}\,C_k. \tag{5}$$

Here $\Gamma(z)$ is the Gamma function and $c_E \approx 0.577$ is the Euler constant.

The quantity C_k is the contribution of the Feynman diagram for the vertex part, in which all the points are joined by an equal number of lines

$$C_k = \int \prod d^2 x_i \prod_{i<j} |x_i - x_j|^{-\frac{4}{k}} \delta^2\left(\frac{\sum x_i}{k}\right) \delta\left(\frac{\sum \ln(x_i^2)}{k}\right) \tag{6}$$

The second significant simplification that is reached by taking the limit in (4) is connected with the two-dimensional character of the Feynman integrals (6), which enables us to obtain the answer in closed form

$$C_k = \frac{\pi^{k-1}}{\Gamma(k-1)} \left\{ \frac{\Gamma\left(\frac{1}{k}\right)}{\Gamma\left(1-\frac{1}{k}\right)} \right\}^k \tag{7}$$

It is easy to verify that the perturbation-theory series (5) for $\psi(g)$ is asymptotic $(C_k(n)|_{k\to\infty} \sim (k!)^{n/2-1})$. It becomes necessary therefore to find a function whose expansion in powers of k coincides with (5), a procedure which is not single-valued. It can be verified with simple models that the correct prescription for summing the series (5) is to replace it by a Watson-Sommerfeld integral

$$\psi(g) = \frac{1}{-2i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{dk}{\sin \pi k} g^k C_k(n), \qquad 1 < \sigma < 2. \tag{8}$$

In the form (8), the function $\psi(g)$ is defined for all g. In particular,

$$\psi(g)|_{g\to 0} = g^2 \frac{\sqrt{2}}{2\pi} \left(\frac{e}{\pi}\right)^{\frac{n}{2}} e^{-(1+c_E)}, \tag{9a}$$

$$\psi(g)\Big|_{g\to\infty} = -\frac{1}{2\pi\sqrt{n}} \frac{g}{(\ln g)^{3/2}}. \tag{9b}$$

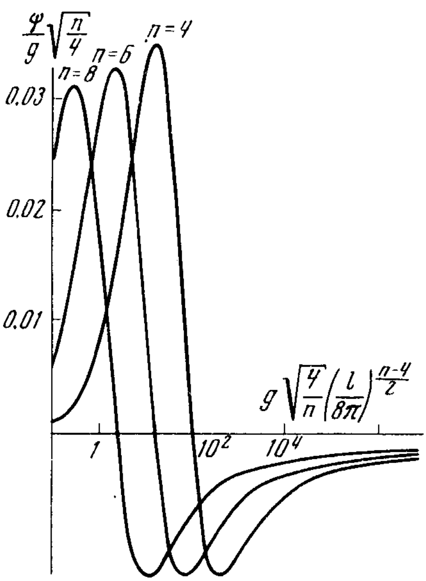

FIG. 1.

Thus, the Gell-Mann—Low function reverses sign with increasing g—there is an ultraviolet stability point $\psi(g_0)=0$ such that $g(p^2/\mu^2, g_\mu)|_{p^2\to\infty}\to g_0$. It can be verified that with increasing n the number of ultraviolet stability points increases like $n^{1/4}$. The figure shows the GML function for two cases of physical interest, $n=4$ and $n=6$. It can be assumed that the true GML function for the theories with $H_{int}=g\int(\phi^4/4!)\,d^4x$ and $H_{int}=g\int(\phi^6/6!)\,d^3x$ will differ from that calculated in the present paper (see the figure) by corrections of the order of $1/n$ (i.e., 1/4 and 1/6 for the two theories indicated above).

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