Modified equation of a nonlinear string and inelastic interaction of solitons

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It is shown that, on going from the equations of the nonlinear string to its correct modification, the soliton interactions become inelastic, and the inelasticity coefficient increases with increasing amplitude of the gliding solitons.

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Recently the Korteweg—de Vries (KdV) equation and the (Boussinesq) nonlinear-string equation

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxxx} \tag{1}$$

have attracted much interest because, first, they describe a large group of nonlinear wave phenomena and, second, they are exactly solvable models.[1] For Eq. (1), N-soliton solutions were obtained[2] as well as an infinite set of commuting integrals of motion, thus demonstrating clearly that this equation is fully integrable[3]; the integrability of the KdV equation was proved rigorously.[4] Randomization is possible, in the nonlinear systems simulated by these equations, as is most clearly pronounced in absolutely elastic interactions of solitons.

It must be emphasized that the KdV equation and (1) are approximate and were obtained for waves ($k \ll 1$). As $u \to 0$, Eq. (1) leads to the dispersion relation $\omega^2 = k^2(1-k^2)$. Thus, Eq. (1) describes an unphysical instability of short waves, $k > 1$, and the Cauchy problem for this equation is incorrect. It can be "regularized" by linearizing the system of equations for ion-sound waves in a plasma (see, e.g., [5]), which leads to the equation

$$Lu = \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} \mp \frac{\partial^4}{\partial x^2 \partial t^2}\right)u = 0.$$

Retaining the form of the nonlinear term, we obtain the correct analog of the nonlinear string equation (the modified Boussinesq equation[6]):

$$u_{tt} = u_{xx} + (u^2)_{xx} + u_{xxtt}. \tag{2}$$

For the replacement of u_{xxxx} by u_{xxtt} in other problems we note that in (1) the terms $(u^2)_{xx}$ and u_{xxxx} are, at small amplitudes $A \ll 1$ and at $k \ll 1$, corrections in comparison with the terms u_{tt} and u_{xx}, and they are equal to each other in zeroth order in k and A. The operator L is obtained in addition by a transition, which is obvious at $k \ll 1$, from $\omega^2 = k^2(1-k^2)$ to $\omega^2(1+k^2) = k^2$. It is clear that the dynamics of long-wave processes remains practically unchanged if the modification (2) is used in place of Eq. (1). Thus, Eq. (2) is convenient for approximate simulation, including computer simulation, of the dynamics of different nonlinear waves with weak dispersion, the spectrum of which at $k \ll 1$ and $u \to 0$ is $\omega^2 \approx k^2(1-k^2)$.

We write (2) in the form of a system, by introducing the "velocity" v:

$$\frac{\partial}{\partial t}\left(1 - \frac{\partial^2}{\partial x^2}\right)u = -\frac{\partial v}{\partial x}, \qquad \frac{\partial v}{\partial t} = -\frac{\partial(u+u^2)}{\partial x}. \tag{3}$$

If $u \to 0$ as $|x| \to \infty$, we obtain from (3) the conservation laws

$$B_1 = \int_{-\infty}^{\infty} u\,dx = \text{const}, \qquad B_2 = \int_{-\infty}^{\infty} v\,dx = \text{const}. \tag{4}$$

Equation (2) has solutions in the form of solitons

$$u = A\,\text{sch}^2\left[\left(\frac{A}{6}\right)^{1/2}\frac{(x-Mt)}{M}\right], \qquad M = \pm\left(1 + \frac{2}{3}A\right)^{1/2}, \tag{5}$$

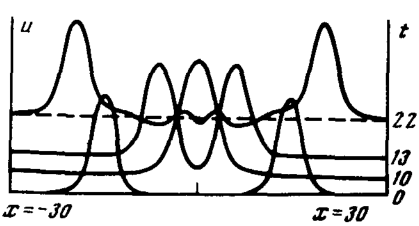

FIG. 1. Inelastic head-on collision of solitons (5) of Eq. (2) at $A = 2$.

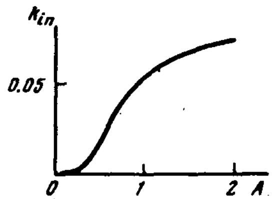

FIG. 2. Dependence of the inelasticity coefficient k_{in} on the amplitude of the colliding solitons (5).

which at $A \ll 1$, having a width $\Delta x \sim (6/A)^{1/2} \gg 1$, hardly differ from the solitons of Eq. (1). The difference between the soliton solutions and their dynamics within the framework of Eqs. (1) and (2) becomes significant at $A \sim 1$, when the relative role of the short waves becomes noticeable in accordance with the for formula $u(k) \sim k\,\mathrm{cosech}[k\pi/(2A/3)^{1/2}]$, which was obtained for the solitons of Eq. (1).

An analysis of the interaction of the solitons, if the latter exist in the considered nonlinear system, is an effective method of determining whether linear randomization is possible in this system. It is impossible, however, to investigate analytically collisions of solitons within the framework of (2). Computer experiments were therefore performed with identical colliding solitons (5) at different amplitudes A. It is seen (Fig. 1) that their collisions are inelastic. We define the inelasticity coefficient k_{in} as the ratio of the amplitude of the non-soliton perturbations produced in soliton collisions to their amplitude A. As $A \to 0$, we have $k_{in} \to 0$, as expected. With increasing A, k_{in} increases monotonically, reaching $k_{in} \sim 0.075$ at $A = 2$ (Fig. 2). Thus, by using Eq. (2) for a more consistent description of short waves, we simulate a nonlinear system in which a redistribution of the energy over the degrees of freedom is possible. From the point of view of the Fermi-Pasta-Ulam nonlinear randomization problem,[7] interest attaches to the indication obtained in the present paper that the rate of randomization increases in this system with increasing amplitude of the nonlinear perturbations.

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