Singular points of a force-free magnetic field

S. I. Syrovatskiĭ

P. N. Lebedev Physics Institute, USSR Academy of Sciences
(Submitted July 2, 1976)
Pis'ma Zh. Eksp. Teor. Fiz. **24**, No. 6, 346–348 (20 September 1976)

PACS numbers: 52.25.–b, 95.50.+y

In the external atmospheres of stars and planets, owing to the more rapid (usually barometric) decrease of the pressure P in comparison with the power-law decrease of the multipole components of the magnetic field $\mathbf{B}$, the condition $P \ll B^2/8\pi$ is usually satisfied, i.e., the plasma parameter $\beta = 8\pi P/B^2$ is small. Under this condition, the state of a plasma with a magnetic field is well described by the zeroth approximation in β, when the pressure P is disregarded completely. In the static case, this corresponds to a force-free magnetic field[1]

$$[\mathbf{B} \times \mathrm{rot}\, \mathbf{B}] = 0, \quad \mathrm{div}\, \mathbf{B} = 0 \tag{1}$$

with currents that fall along the magnetic force lines.

On the other hand, it is precisely in the outer atmospheres that one observes such specific flare processes in plasma as solar and stellar flares, magnetospheric substorms, or accelerator processes in the magnetosphere of Jupiter. In the simplest case of a planar two-dimensional force-free field, when the field is simply potential, this flare process is attributed to a neutral current layer (for details see[2]). Such layers, as shown for a two-dimensional field in[3] appear in a well-conducting plasma at the location of singular zero points of the magnetic field, i.e., points at which $\mathbf{B} = 0$ but the electric field $\mathbf{E}$ is not equal to zero by virtue of the boundary conditions.

It will be shown below that this result remains in force for a force-free field of general form. Namely, in an arbitrary three-dimensional force-free magnetic field the points at which $\mathbf{B} = 0$, but $\mathbf{E} \neq 0$ by virtue of the boundary conditions, are singular in the sense that the magnetohydrodynamic problem of plasma flow in the vicinity of these points has in general no continuous solutions. In analogy with the two-dimensional case, it is natural to assume that discontinuity surfaces—current layers—should arise at these points. We emphasize that we are dealing throughout with an approximation in which the conductivity is assumed to be ideal, an approximation sufficiently well satisfied by both cosmic plasma and by fast processes in a laboratory plasma.

For the proof it suffices, in fact, to show that points with the indicated properties can indeed exist. We present this proof for the case of adiabatically slow deformations of the force-free field, when

$$\epsilon = v_0 / v_A \ll 1, \tag{2}$$

where v_0 is the characteristic velocity of the plasma and $v_A = B/\sqrt{4\pi\rho}$ is the Alfven velocity.

We choose as the units of length, velocity, time, density, and magnetic and electric field intensities the values

$$R_o,\ v_o,\ t_* = R_*/v_o,\ \rho_o,\ B_o,\ v_oB_o/c. \tag{3}$$

We then obtain at $\beta = 0$ the dimensionless equations of the problem in MHD approximation:

$$\epsilon^2 d\mathbf{v}/dt = \rho^{-1}[\mathrm{rot}\,\mathbf{B} \times \mathbf{B}], \tag{4}$$

$$\partial\mathbf{B}/\partial t = -c\,\mathrm{rot}\,\mathbf{E},\quad \mathrm{div}\,\mathbf{B} = 0, \tag{5}$$

$$\mathbf{E} = -[\mathbf{v} \times \mathbf{B}]/c, \tag{6}$$

$$\partial\rho/\partial t + \mathrm{div}\,\rho\mathbf{v} = 0, \tag{7}$$

where $d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v}$. Expanding all the quantities in powers of ϵ^2, for example $\mathbf{v} = \mathbf{v}_0 + \epsilon^2\mathbf{v}_1 + \cdots$, we write down the equations containing the zeroth-approximation terms. From among these, only the equation

$$\rho_o \frac{d\mathbf{v}_o}{dt} = [\mathrm{rot}\,\mathbf{B}_1 \times \mathbf{B}_o] + [\mathrm{rot}\,\mathbf{B}_o \times \mathbf{B}_1] \tag{8}$$

contains the terms of the next higher approximation (the terms on the right-hand side). These, however, are easily eliminated by taking the scalar product of (8) and B_0. As a result we obtain a closed system of equations for the zeroth-approximation quantities:

$$[\mathbf{B}_o \times \mathrm{rot}\,\mathbf{B}_o] = 0,\quad \mathrm{div}\,\mathbf{B}_o = 0, \tag{9}$$

$$\mathbf{B}_o\, d\mathbf{v}_o/dt = 0, \tag{10}$$

$$\mathbf{E}_o + [\mathbf{v}_o \times \mathbf{B}_o]/c = 0, \tag{11}$$

$$\mathrm{rot}\,\mathbf{E}_o = -\frac{1}{c}\frac{\partial\mathbf{B}_o}{\partial t}, \tag{12}$$

$$\frac{\partial\rho_o}{\partial t} + \mathrm{div}\,\rho_o\mathbf{v}_o = 0. \tag{13}$$

Equation (9) determines the vector of the equilibrium magnetic field $\mathbf{B}_0$. As shown in[4], the problem of determining the force-free field from specified values of the field vector at the boundary of the region has a unique solution subject to certain restrictions on the permissible boundary values of the vector $\mathbf{B}_0$ (conditions at boundary points that are connected by one and the same force line). The solution of (9) determines $\mathbf{B}_0 = \mathbf{B}_0(\mathbf{r}, t)$, where the time dependence is parametric, in terms of the boundary conditions.

Equation (10) determines the velocity component along the force line (there is no acceleration in this direction), while Eq. (11) determines the transverse component in terms of the known $\mathbf{B}_0$ and $\mathbf{E}_0$. As already mentioned, the field $\mathbf{B}_0$ is determined independently of Eq. (9). Equation (12) determines then the solenoidal part of the electric field $\mathbf{E}_0$. It is important in what follows that the potential part of the field $\mathbf{E}_0$ is determined by independent boundary conditions, namely the distribution of the normal component of the vector $\mathbf{E}$ or the corresponding density of the electric charge on the boundary of the region.

Finally, the plasma density is expressed in terms of the known velocity $\mathbf{v}_0$ by Eq. (13).

The subsequent reasoning is perfectly analogous to that of [3]. Namely, the boundary conditions for the fields $\mathbf{B}_0$ and $\mathbf{E}_0$ admit, in the general case, of the existence of points at which $\mathbf{B}_0 = 0$ (null points) inside the region, including sites at which $\mathbf{E}_0 \neq 0$ (singular null points). The latter follows from the above-noted independence of the potential part of electric field $\mathbf{E}_0$. However, the existence of singular null points contradicts Eq. (11). This means that the system (8)—(13) has no continuous solutions in the vicinity of the singular null points. In order to obtain the solution of the boundary-value problem we can proceed in two ways: either go outside the framework of the employed approximation by taking into account the smoothed-out dissipative terms, or else, preserving the ideal-medium approximation, introduce corresponding discontinuity surfaces. On these surfaces, such physical quantities as the density of the plasma and the vectors of the magnetic field and velocity can undergo discontinuities, while remaining bounded. In this respect, there is full analogy with ordinary hydrodynamics, in which the appearance of discontinuities in an ideal medium corresponds physically to the need for taking into account dissipative terms if a continuous solution is to be obtained.

Obviously, by virtue of the requirement that the magnetic field be bounded, the singularities in a force-free field cannot have the character of linear currents. Thus, just as in the two-dimensional case, discontinuity surfaces, or current layers, should appear in the place of the singular null points of the force-free magnetic field.

[1]S. Lundquist, Ark. Fys. **2**, 361 (1950).
[2]S. I. Syrovatskiĭ, Izv. Akad. Nauk SSSR Ser. Fiz. **39**, 359 (1975).
[3]S. I. Syrovatskiĭ, Zh. Eksp. Teor. Fiz. **60**, 1727 (1971) [Sov. Phys. JETP **33**, 933 (1971)].
[4]M. M. Molodenskiĭ, Astron. Zh. **45**, No. 4 (1968) [Sov. Astron. **12**, No. 4 (1969)].