Oscillating particle-like solutions of the nonlinear Klein-Gordon equation

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A denumerable set of oscillating spherically-symmetrical particle-like solutions of the Klein-Gordon equations with cubic nonlinearity has been obtained. The simulation-capable or extended particles turn out to be weakly-radiating and long-lived.

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In the last decades, numerous attempts were made to find particle-like solutions (PLS) of the relativistically-invariant nonlinear field equations (see, e.g., the review[1]).

In the present paper we confine ourselves to scalar real fields satisfying the Klein—Gordon equation with cubic nonlinearity:

$$u_{tt} - \Delta u + u - u^3 = 0. \tag{1}$$

Equation (1) has nontrivial steady-state solutions—planar one-dimensional solitons[2] and spherically symmetrical (ss) PLS.[3,4] These solutions are, however, unstable.[5,6] On the other hand, within the framework of Eq. (1) in the (x,t) case $(\Delta \to \partial^2/\partial x^2)$, stable[1)] self-localized nonlinear oscillations[9] (which we shall call for brevity "pulsons") can be analytically described.

Of great interest in elementary-particle physics are, of course, spatial PLS. The first example of long-lived *ss* pulsons was observed in[10] as a result of an investigation of the equation for the Higgs field. Their amplitude $c(t)$ decreases slowly as a result of weak radiation, and the lifetime is $\tau \sim 10^3$. In the present paper, using the Fourier method in the presence of a small parameter $(u^2 \ll 1)$ [9] and a computer, we obtain and investigate the *ss* pulsons of Eq. (1).

We seek the solution of (1) in the form

$$u(r, t) = a(r)\cos\omega t + b(r)\cos 3\omega t + \dots . \tag{2}$$

Substituting (2) in (1), we arrive at the nonlinear eigenvalue problem

$$\begin{gathered} a_{rr} + \frac{2}{r}a_r + \frac{3}{4}a^3 = \lambda a, \qquad \lambda = 1 - \omega^2, \\ a_r(0) = 0, \qquad a(\infty) = 0. \end{gathered} \tag{3}$$

Let $y(r)$ be its solution at $\lambda = 1$. It is easy to verify that then $y_\lambda = \sqrt{\lambda}\, y(\sqrt{\lambda} r)$ is the solution of (3) for a given $\lambda = 1 - \omega^2$. We introduce $A = \sqrt{\tfrac{3}{4}}\,a$. The equation obtained for the variable A

$$A_{rr} + \frac{2}{r}A_r - A + A^3 = 0 \tag{4}$$

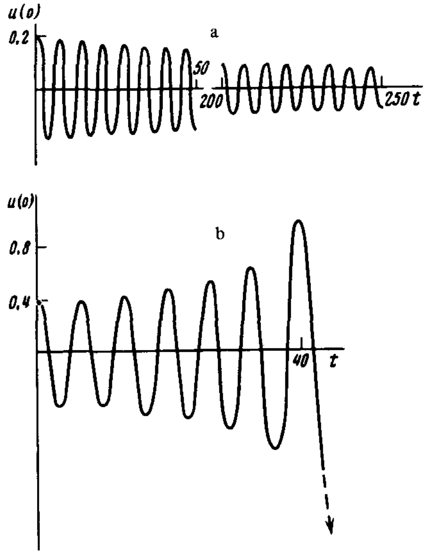

FIG. 1. Plot of $u(0,t)$ for the field function $u(r,0)$ given by formula (5); a—$k=1.3$, b—$k=0.8$.

under the boundary condition $A_r(0)=0$, $A(\infty)=0$ have a denumerable set of solutions $A_i(r), i=1,2,\ldots,n,\ldots$, with the ith solution having $(i-1)$ zeroes; $A_1(0) \approx 4.34 < A_2(0) \approx 14.10 < A_3(0) \approx 29.13 < \cdots < A_n(0) < \cdots$. [3,4]

Thus, the functions

$$u_i(r,t) = \sqrt{\frac{4}{3}}\, u_0 A_i(k u_0 r) \cos(\sqrt{1-u_0^2}\, t) = u_m \frac{A_i(k u_0 r)}{A_i(0)} \cos(\sqrt{1-u_0^2}\, t), \quad k=1 \tag{5}$$

with accuracy on the order of $u_m^2 \ll 1$ are solutions of Eq. (1) and describe *ss* pulsons. The expression for $b(r)$ at $u_m^2 \ll 1$ can be easily obtained:

$$b(r) = -\frac{1}{12\sqrt{3}} u_0^3 A_i^3(u_0 r). \tag{6}$$

The dynamics of the PLS (5) was investigated with the aid of a computer. We considered the first three modes of the solutions ($i=1,2,3$) at amplitudes $u_m=0.2$, 0.4, and 0.7. At $u_m \leq 0.4$ the results of the calculations are approximated with high accuracy by formula (5) (deviation less than 1%).

We note in particular that, at any rate when $u_m^2 \ll 1$, the radiation of the pulson at the intensity is very small, and its lifetime $\tau \to \infty$ as $u_m{}^2 \to 0$. If the larger value $u_m=0.7$ is specified in (5), then the pulsation amplitude $c(t)$ decreases slowly to $c(t)=0.63$ by the instant $t=80$, and the characteristic radius of the pulson R_c increases.

The field cluster obtained by compression of (5) along the r axis ($k>1$) at a fixed amplitude spreads out gradually, so that $R_c \to \infty$ as $t \to \infty$, while $c(t)$ decreases monotonically (Fig. 1a). Conversely, a cluster that is wider than the pulson (5) ($k<1$) begins to contract towards the center, and $c(t)$ increases (the slower the closer k to unity) to a value $u_{ci} \sim 1$. This is followed by an "explo-

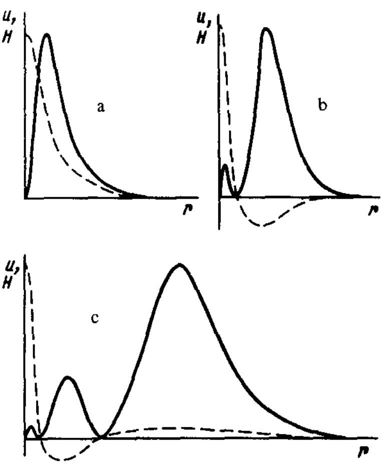

FIG. 2. Structures of the first three modes of the pulsons of (5): a) $i=1$, b) $i=2$, c) $i=3$. Dashed line—the function $u_i(r,0)=\sqrt{\tfrac{4}{3}}u_0A_i(u_0,r)$, solid—plots of $\mathcal{H}_i(r)$.

sive" (faster than exponential) formation of a field singularity, $|u(0,t)|\to\infty$ (Fig. 1b). It is possible that this effect is due to the shape of the "potential energy" curve of the field $U(u)=u^2-(u^4/2)$. It appears that the amplitude of the pulsons of Eq. (1) $u_{\max}$ is bounded from above by a constant $u^* \sim 1$: $u_{\max}<u^*\sim 1$. To describe these pulsons at $u_{\max}^2 \lesssim 1$ it is necessary to take into account the next terms of the expansion $a(r), b(r), \ldots$ in powers of $u_{\max}$.[9] The energy of the pulsons (5)

$$E=\frac{1}{2}\int_0^\infty [u_r^2+u_t^2+u^2-(u^4/2)]r^2dr=\int_0^\infty Hr^2dr=\int_0^\infty \mathcal{H}dr \tag{7}$$

is conveniently calculated for the instant when $u_t=0$; substituting (5) in (7) we have for the ith mode

$$E_i=I_1^{(i)}(u_0)+I_2^{(i)}(u_0)-I_3^{(i)}(u_0)=u_0(I_1^{(i)}-I_3^{(i)})+u_0^{-1}I_2^{(i)},$$

$$I_1^{(i)}=\frac{1}{2}\int_0^\infty\left(\frac{dy_i}{dr}\right)^2 r^2dr;\quad I_2^{(i)}=\frac{1}{2}\int_0^\infty y_i^2r^2dr;\quad I_3=\frac{1}{4}\int_0^\infty y_i^4r^2dr. \tag{8}$$

In the limit as $u_0\to 0$ we have $E_i\approx u_0^{-1}I_2^{(i)}$, and the main contribution to the energy density of the field $H(r,t)$ is made by the terms u_t^2 and u^2, the sum of which, $u_t^2+u^2=u_i^2(r,0)\times(\cos^2\omega t+\omega^2\sin^2\omega t)$, is constant accurate to $\sim u_0^2$ for each r, by virtue of the fact that $\omega^2=(1-u^2)\to 1$ as $u_0^2\to 0$. Therefore the distributions $H(r)$ and $\mathcal{H}(r)$ (Figs. 2a,b,c) are independent of the time at the same degree of accuracy. We note that since $u_0=u_m/\sqrt{\tfrac{4}{3}}A_i(0)$, the distribution of the pulson "mass" along the radius at a given amplitude u_m is conserved in time more accurately the larger the number of the mode i.

Thus, by forgoing the requirement that the field function $r(r,t)$ be stationary, we can construct a denumerable set of PLS of Eq. (1), which are single-field models of long-lived particles with zero spin. In the limit as $u_0\to 0$, at equal

u_0, the masses of these particles $m_k = E_i$ are related like $I_2^{(i)}$, and at identical u_m they are related like $I_2^{(i)}/A_i(0)$ ($\approx 1:2:3:4:9:\cdots$). It is possible that similar oscillating solutions will prove to be useful for the description of ψ bosons (a soliton model for these particles was first proposed in[11], where a one-dimensional equation for the Higgs field was considered).

The results obtained at $u^2 \ll 1$ can be directly applied to the case of the sine-Gordon equation

$$u_{tt} - \Delta_{rr} u + \sin u = 0. \tag{9}$$

However, within the framework of (9), unlike in (1), long-lived pulsons with amplitude $c(t) > 1, c(t) \sim 2\pi$ have been obtained. We note that the pulsons of[10] can also be described at amplitudes $c(t) \ll 1$ by the Fourier method.

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1) In the numerical experiment we observed formation of flat pulsons out of oscillating field clusters close to them.

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