Line and point singularities in superfluid He^3

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A classification is presented for the topologically stable singularities in the *A* and *B* phases of He^3, with the spin-orbit interaction taken into account.

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The first to consider singularities in superfluid He^3 was de Gennes.[1] He has pointed out that the *A* phase of He^3 can contain singularities of the type of vortex lines, disgyrations (linear singularities in the field of the vector **l** characterizing the direction of the orbital momentum of the pair), and singularities in the field of the vector **V** that characterizes the spin state. A number of papers have by now been devoted to various singularities in the *A* and *B* phases,[2–4] including singularities of monopole character.[5–7] In this paper we present a topological analysis that helps answer the following questions: 1) what types of singularities are topologically stable, i.e., cannot be transformed into a nonsingular configuration by means of a continuous deformation? 2) Which of them are topologically equivalent, i.e., can go over into one another via a continuous deformation? 3) What happens when singularities coalesce? The topological singularity analysis presented in[4,5] is incorrect. Our analysis, which is based on the use of the so called homotopic groups (see, e.g., [8]), makes it possible to set each line singularity and each point singularity in correspondence with an element of a homotopic group π_1 and π_2, respectively. Thus, the classification of the singularities reduces to an identification of the particular groups for a given type of order parameter. The following rules hold in this case: 1) if two singularities correspond to the same homotopic-group element, then they can be converted into each other by a continuous deformation; 2) if a single homotopic-group element corresponds to a singularity, then this singularity is homotopically unstable; 3) if a singularity characterized by element *a* of a group coalesces with a singularity characterized by element *b*, the result is a singularity corresponding to the element $a+b$. That is to say, the coalescence of singularities corresponds to group addition of the elements of the homotopic group. Since in our case all the homotopic groups are Abelian with a finite number of generators, we can characterize each singularity by a set of integer indices.

The order parameter in the *A* and *B* phase takes the respective form

$$A_{ik} = \text{const}\, V_i\,(\Delta'_k + i\Delta''_k), \quad A_{ik} = \text{const}\; e^{i\Phi} R_{ik}(\vec{a}) \,, \tag{1}$$

where $\mathbf{V}$, $\mathbf{\Delta}'$, and $\mathbf{\Delta}''$ are unit vectors, with $\mathbf{\Delta}'\cdot\mathbf{\Delta}''=0$ and $\mathbf{l}=\mathbf{\Delta}'\times\mathbf{\Delta}''$; Φ is the

The presence of spin-orbit interaction gives rise, besides the usual co-phase of the condensate, and R_{ik} is the matrix of rotation through an angle $|\boldsymbol{\alpha}| \leq \pi$ about the $\boldsymbol{\alpha}$ axis.
herence length $\xi(T)$, to a new length $R_c \sim (10^2-10^3)\xi(T)$ (see[9]), over which the spin-orbit interaction becomes comparable with the gradient energy. Therefore the classification of the singularities depends on the dimensions of the investigated region. If one of the characteristic dimensions of the region is $\xi \ll r \lesssim R_c$, then spin-orbit interaction can be neglected in this region. In regions with dimensions exceeding R_c, the spin-orbit interaction alters the structure of the order parameter, fixing the rotation angle $|\alpha| \approx 104°$ in the B phase and orienting $\mathbf{V} \parallel \mathbf{l}$ in the A phase, and changes by the same token the types of the singularity.

We now list the types of singularities with a brief indication of the homotopic-group elements to which they correspond.

I. *B phase at* $r \lesssim R_c$. There are no point singularities. The line singularities are the following (we shall describe the line singularities by a cylindrical coordinate system (z, ρ, φ) with the z axis along the singular line and $0 \leq \varphi \leq 2\pi$): 1) Vortices with N circulation quanta $\Phi(r) = N\varphi$, where N is an arbitrary integer. They are completely analogous to the vortices in He II. When the vortices coalesce, addition of the circulation quanta takes place just as in He II. 2) Line singularities in a field R_{ik}, characterized by an integer index m equal to 0 or 1, with $m = 0$ corresponding to absence of a singularity. When two singularities with $m = 1$ coalesce, the indices are added in modulo 2, i.e., $1 + 1 = 0$, and a nonsingular configuration is obtained. A similar situation takes place also in nematic liquid crystals, where there is only one type of topologically stable singularities (with a Frank index $m = 1$), and all others can be reduced either to this type of singularity, or to a nonsingular configuration (see[10]). The singular solution with $m = 1$ takes the form $\boldsymbol{\alpha}(\mathbf{r}) = (\phi - \pi)\hat{\mathbf{z}}$.

The classification given above is a consequence of the fact that the region of variation of the order parameter in the B phase is $R = S^1 \times SO_3$. The homotopic group for this manifold are $\pi_1(R) = Z + Z_2$ and $\pi_2(R) = 0$ (the latter means that π_2 is trivial; Z is the group of integers and Z_2 is the group of residues in modulo 2).

II. *B phase with* $r \gg R_c$. (Here $R = S^1 \times S^2$, $\pi_1(R) = Z$, $\pi_2(R) = Z$). We have the following: 1) the vortices $\Phi(\mathbf{r}) = N\phi$ considered above; 2) point singularities in the field of the vector $\omega = \boldsymbol{\alpha}/|\alpha|$. These singularities are characterized by the whole-number invariant

$$N = \frac{1}{8\pi} \int dS_i \, e_{ijk} \left(\vec{\omega} \left[\frac{\partial \vec{\omega}}{\partial x_j} \, \frac{\partial \vec{\omega}}{\partial x_k} \right] \right), \tag{2}$$

where the integration is over the surface surrounding the singular point (N is the degree of mapping of this surface on the sphere $|\omega| = 1$ and runs through all the integers). When singularities coalesce, the invariants are added. A singularity with $N = 1$ is a "hedgehog" $\omega = \mathbf{r}$ (where r, θ, and φ are spherical coordinates). The radius of the core of the "hedgehog" is of the order of R_c, and the field $\boldsymbol{\alpha}(r)$ near the "hedgehog" takes the form $\boldsymbol{\alpha} = f(r)\hat{\mathbf{r}}$, where $f(r) \to 104°$ at $r \gg R_c$ and $f \to 0$ as $r \to 0$.

III. *A phase at* $r \lesssim R_c$. (Here $R = (S^2 \times SO_3)/Z_2$, $\pi_1(R) = Z_4$, $\pi_2(R) = Z$). We have: 1) singular points characterized by the whole-number invariant (2), in which ω must be replaced by $\mathbf{V}$. These singular points are analogous to the singular points in nematic liquid crystals (the role of the director is played by $\mathbf{V}$). Just as in nematic crystals, the singular points with $N = \pm\, |N|$ are indistinguishable;

however, if there are several singular points, then each is characterized by a definite sign, accurate to a common sign. 2) Three types of singular lines, characterized by an index m that takes on the values 0, 1, 2, and 3. When these lines coalesce, addition of the indices in modulo 4 takes place. The singular line with $m=1$ is a vortex with a circulation quantum $1/2$, on which is superimposed the disclination of the vector $\mathbf{V}$ with a unit Frank index ($\mathbf{\Delta}'+i\mathbf{\Delta}''=e^{i\phi/2}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})$, $\mathbf{V}=\mathbf{x}\cos(\phi/2)-\hat{\mathbf{y}}\sin(\phi/2)$. The singular line with $m=3$ differs from $m=1$ only in that the circulation quantum is equal to $-1/2$. The singular lines with $m=2$ constitute either a vortex with a unit circulation ($\mathbf{V}=\text{const}$, $\mathbf{\Delta}'+i\mathbf{\Delta}''=e^{i\phi}(\hat{\mathbf{x}}+i\hat{\mathbf{y}})$), or a disgyration, for example ($\mathbf{V}=\text{const}$, $\mathbf{l}=\pm\boldsymbol{\rho}$, $\mathbf{\Delta}'=\boldsymbol{\phi}$, $\mathbf{\Delta}''=\pm\hat{\mathbf{z}}$), the latter having the lowest energy among all the singularities of this type. We note that since $2+2=0$ a vortex with two circulation quanta goes over continuously into a nonsingular configuration.

IV. *A phase at* $r\gg R_c$. (Here $R=SO_3$; $\pi_1(R)=Z_2$, $\pi_2(R)=0$.) There are no point singularities. The line singularities can be characterized by the same index m as in III, except that it now takes on the values 0 and 2. Among the singular lines with $m=2$, the maximum energy is possessed by the vortex with unit circulation. At $p\lesssim R_c$ it goes over into a disgyration or into two singularities with $m=1$ (or $m=3$), spaced $\sim R_c$ apart.

A few words now on monopoles (or vortons as they are called in[5]). Contrary to the statement made in[5], there is no topological invariant for vortons, so that their existence and stability depend on the possible potential barrier that is produced when they are transformed into a different configuration. It can be shown that at $r\lesssim R_c$ in the A phase the vortons, together with the two vortices that emerge from it, belong to the type of singular lines with $m=2$ and can relax without a barrier into a stable disgyraton with $\mathbf{l}=\hat{\boldsymbol{\rho}}$. In this region, however, there is a stable point singularity ($\mathbf{V}=\text{const}$, $\mathbf{l}=\hat{\boldsymbol{\theta}}$, $\mathbf{\Delta}'=\hat{\boldsymbol{\phi}}$, $\mathbf{\Delta}''=\hat{\mathbf{r}}$ are the unit vectors of the spherical coordinate system), which connects a disgyration with $\mathbf{l}=\hat{\boldsymbol{\rho}}$ and a disgyration with $\mathbf{l}=-\boldsymbol{\rho}$. This singularity also belong to the type $m=2$, but cannot relax into a stable disgyration without a barrier. This singularity ceases to be stable in the region $r\gg R_c$.

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