# New example of quantum-mechanical problem with hidden symmetry

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(Submitted June 11, 1976)
Pis'ma Zh. Eksp. Teor. Fiz. **24**, No. 4, 196–199 (20 August 1976)

PACS numbers: 03.65.Ge, 03.65.Kh

There are several known quantum-mechanical problems with hidden symmetry,[1)] such as the oscillator, the Kepler problem, the rotator, and a few others, which do not admit of a direct physical interpretation. The hydrogen atom, for example, besides the usual rotational symmetry, has a $O(4)$ symmetry (fraction of the discrete symmetry) due to the existing of the conserved Runge-Lenz vector.[2]

  

In an analysis of problems in which the degrees of freedom interact with an inhomogeneous field, we have observed an example of such a system with dynamic symmetry. Namely, a neutron in the magnetic field of a linear current forms bound states whose spectrum is determined by the dynamic symmetry group $O(3)$. The level energies are determined by the quantum number $n$:

$$E_n = -\frac{1}{n^2}(1.7\cdot 10^{-6}J^2)\,\text{eV}, \tag{1}$$

where $J$ is the current in amperes.

We start with a direct solution of the Schrödinger equation. A neutral nonrelativistic spin-1/2 particle magnetic moment $\mu$ is described by the Hamiltonian

$$H = \frac{p^2}{2M} - \mu\vec{\sigma}\vec{\mathcal{H}}, \tag{2}$$

where $\vec{\mathcal{H}}$ is the external magnetic field. We consider the field produced by a linear current of strength $J$, directed along the $z$ axis

$$\vec{\mathcal{H}} = 0.2J\left(\frac{y}{r^2}, -\frac{x}{r^2}, 0\right). \tag{3}$$

if $r$ is in centimeters and the current in amperes, then $\mathcal{H}$ is in guasses. For this configuration, the motion along the $z$ aixs is free and we can immediately eliminate that part of the Hamiltonian which is connected with this motion. It is convenient to change over to dimensionless variables by means of the scale transformation

$$\kappa \mathbf{r} = \vec{\rho}, \qquad \mathbf{p}/\kappa\hbar = \vec{\pi}, \quad \text{where} \quad \kappa = \sqrt{-2ME/\hbar} \tag{4}$$

We focus our attention on the discrete spectrum. As a result, the equation takes the form

$$[\vec{\pi}^2 + 1 + 2n(\sigma_x y - \sigma_y x)^{-1}]\psi = 0, \tag{5}$$

where $n = -0.2\,\mu MJ/\kappa\hbar$,[2] and $x$ and $y$ denote the projections of the dimensionless vector.

We describe briefly the solution scheme, the details of which will be presented in a separate paper. Multiplying the equation by the quantity $(\sigma_x y - \sigma_y x)$ and changing over to the momentum representation, we obtain a system of first-order equations for the components of the spinor $\psi$. Separating the variables in polar coordinates and eliminating one of the spinor components, we obtain a second-order differential equation that reduces, through some change of variables, to the equation for the hypergeometric function. As a result of the quantization condition, the number $n$ defined in the explanation of Eq. (5) takes on positive integer values. In the momentum representation, the discrete-spectrum wave functions are

$$
\psi(p) = C_{n,m}\begin{pmatrix} \dfrac{(\pi_x + i\pi_y)^{m-1/2}}{1+\vec{\pi}^2}\, F\left(n, -n, m+\tfrac{1}{2}, \dfrac{\vec{\pi}^2}{1+\vec{\pi}^2}\right) \\[2ex] \dfrac{n}{m+1/2}\, \dfrac{(\pi_x + i\pi_y)^{m+1/2}}{(1+\vec{\pi}^2)^2}\, F\left(n+1, -n+1, m+3/2, \dfrac{\vec{\pi}^2}{1+\vec{\pi}^2}\right) \end{pmatrix}, \tag{6}
$$

The level with the quantum number $n$ has $2n$-fold degeneracy. The wave functions with a given energy value can be classified with the aid of the operator for the projection of the total angular momentum on the $z$ axis. Its eigenvalues $m$ range from $-n+1/2$ to $n-1/2$. Formula (6) corresponds to the case of positive $m$.

In analogy with the quantum oscillator and with the Kepler problem, it can be assumed that in our case the additional degeneracy is due to the existence of new integrals of motion.

We rewrite (5) in the form $(K+1)\psi = 0$, where $K$ is, apart from a constant factor, the Hamiltonian of our system, but subjected to the scale transformation (4)

$$
K = \vec{\pi}^2 + 2n(\sigma_x y - \sigma_y x)^{-1}. \tag{7}
$$

We consider the triad of operators

$$
\begin{aligned}
A_x &= -n(\sigma_x y - \sigma_y x)^{-1} y + \tfrac{1}{2}(\pi_x J_z + J_z \pi_x), \\
A_y &= n(\sigma_x y - \sigma_y x)^{-1} x + \tfrac{1}{2}(\pi_y J_z + J_z \pi_y), \\
J_z &= \frac{\sigma_z}{2} + (x\pi_y - y\pi_x).
\end{aligned} \tag{8}
$$

One of them is the projection of the angular momentum on the $z$ axis. The two others satisfy the commutation relations

$$
\begin{aligned}
&[A_x, J_z] = -iA_y \qquad &&[A_x, K] = 0, \\
&[A_y, J_z] = -iA_x \qquad &&[A_y, K] = 0, \\
&[A_x, A_y] = -iKJ_z .
\end{aligned} \tag{9}
$$

Redefining the operators in accord with the formula

$$
\begin{aligned}
J_x &= A_x(-K)^{-1/2}, \\
J_y &= A_y(-K)^{-1/2}
\end{aligned} \tag{10}
$$

we obtain for the discrete spectrum the dynamic symmetry group $O(3)$. The generators of this group and the Hamiltonian $K$ are connected by a relation that can be established by direct calculation

$$K = -\frac{n^2}{J^2 + 1/4} \, . \qquad (11)$$

A similar connection is known also in the Kepler problem [1].

Equation (6) can now be rewritten in the form $(\mathbf{J}^2 + \frac{1}{4} - n^2)\psi = 0$. The properties of the irreducible representations of the rotation group are well enough known. We are interested in representations with half-integer projections. They are characterized by the half-integer positive number $j$. It is easily seen that $n = j + 1/2$.

The continuous-spectrum wave functions form the basis for the representation of the second fundamental series of the group $O(2,1)$. [3]

For problems with hidden symmetry one frequently introduces the so-called non-invariance group, one of the irreducible representations of which describes all the bound states of the system. For the modified "Hamiltonian"

$$\widetilde{H} = (\vec{\pi}^2 + 1)(\sigma_x y - \sigma_y x) \qquad (12)$$

the spectrum of which is linear and whose wave functions correspond to the wave functions of the initial Hamiltonian, we have obtained such a group, which turned out to be isomorphic to the complex form of the $O(5)$ group.

In conclusion, the authors thank S. S. Gershtein for valuable discussions.

1) For a better understanding of the concept of hidden symmetry we refer the reader to Popov's review [1].
2) The sign in the formula for $n$ was chosen with allowance for the fact that the neutron magnetic moment is negative, so that $n$ is positive.

1 In: Fizika vysokikh energiĭ i teoriya élementarnykh chastits (High-Energy Physics and Theory of Elementary Particles), Naukova dumka, Kiev, 1967, p. 702.
2 V. A. Fock, Z. Physik **98**, 145 (1935).
3 N. Ya. Vilenkin, Spetsial'nye funktsii i teoriya predstvleniĭ grupp (Special Functions and Theory of Group Representations), Nauka, 1965, p. 305.