Conservation law in the $(\cos\phi-1)_2$ quantum theory and in the massive thirring model[1)]

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We prove the existence of an infinite number of conservation laws in the massive Thirring model, when $\psi_a(x)$ are elements of a Grassman algebra, and the absence of anomalies in the quantum currents.

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A remarkable fact is valid in quantum theory for the model with Lagrangian $L=\frac{1}{2}(\partial_\mu\phi)^2+m^2/\beta^2(\cos\beta\phi-1)$ [1] and the massive Thirring model[2] namely the conservation, after scattering, of the set of momenta of the particles in the initial state, and factorization of the S matrix[3], if the classical conservation laws remain valid upon quantization. We have ascertained that this is indeed the case and present recurrence relations for the calculation of conserved currents in the Thirring model, when the fields $\psi_a(x)$ are not c numbers[2] but elements of a Grassmann algebra.

1. Local conserved currents for the sine-Gordon equation take a simple form in terms of the variables $\tau=(t+x)/2$, $\sigma=(t-x)/2$ and are calculated from the recurrence relations

$$j_\tau^{(n+1)}=\phi_\tau\partial_\tau(j_\tau^{(n)}/\phi_\tau)+(\beta^2/4)\sum_{k+l=n} j_\tau^{(k)}j_\tau^{(l)}+\phi_\tau^2\delta_{n,0};\quad \phi_\tau\equiv\partial\phi/\partial\tau,$$

$$j_\sigma^{(n+1)}=-2m^2/\beta^2\,\delta_{n,0}\cos\beta\phi+m^2/\beta\,(j_\tau^{(n)}/\phi_\tau)\sin\beta\phi;\quad \partial_\sigma j_\tau^{(n)}+\partial_\tau j_\sigma^{(n)}=0.$$

The currents $j_\tau^{(n)}$ are homogeneous functions in ∂_τ of degree $n+1$, therefore the Green's functions of the currents $j_\tau^{(n)}$ require renormalization in spite of the supernormalizability of the theory. As noted in [4], this can lead to anomalies in the Ward identities (WI) and to violation of the classical conservation laws. Using the quantum equations of motion in the N-product formalism [5] with an effective Lagrangian [6]

$$L = \tfrac{1}{2}:((\partial_\mu\phi)^2 - m^2\phi^2): + :(\tilde{m}^2/\beta^2(\cos\beta\phi - 1) + m^2/2\,\phi^2):,\ \tilde{m}^2 = m^2 + \sum_{k=1}^{\infty}\Delta_k\beta^{2k}$$

for the current

$$j_\tau^{(n)}(\partial_\sigma j_\tau^{(n)} = \sum_{m=0}^{n-1} Y_m\partial_\tau^m(\phi_{\sigma\tau}))$$

we obtain the differential WI

$$< N_{n+2}[\partial_\sigma j_\tau^{(n)}](x)\,X> = -< N_n[\partial_\tau j_\sigma^{(n)}](x)\,X> + m^2\sum_{l=0}^{n-1} < N_n[Y_l(\partial_\tau^l\phi$$

$$-\{\partial_\tau^l\phi\})](x)\,X> - i\sum_{j=1}^{k}\left(\sum_{l=0}^{n-1} < N_{n-l}[Y_l](x)\,\hat{X}>\partial_\tau^l\right)\delta(x-y_j);$$

$$X = \prod_{j=1}^{k}\phi(y_j), \qquad \hat{X} = \prod_{i\neq j}\phi(y_i).$$

where the index a at the sign of the normal product $N_a[B](x)$ determines the order of the subtractions in the subgraphs containing the vertex corresponding to the operator $B(x)$, while the curly brackets correspond to two supersubtractions. Integrating with respect to d^2x we can transform the integral of the anomalous terms

$$\sum_{l=0}^{n-1}\int d^2x < N_n[Y_l(\partial_\tau^l\phi - \{\partial_\tau^l\phi\})](x)\,X> = \sum_{l=1}^{n}\int d^2x < N_n[B_l(\phi - \{\phi\})](x)X>.$$

The monomials B_l are homogeneous of degree n in ∂_τ and of not less than third degree in ϕ. Using the Zimmermann identities and the quantum equations of motion we obtain for the anomalous terms an inhomogeneous system of equations, the determinant of which differs from zero at sufficiently small β. The inhomogeneous terms of the system are Schwinger terms of the differential WI of the form $\langle N_n[B_l](y_j)\hat{X}\rangle$. Thus, in the integral of WI there remain only the Schwinger terms, and on going to the mass shell in the momentum representation we obtain the classical conservation laws $\sum(p_j^{in})^n = \sum(p_j^{out})^n$; $n = 1, 2, \cdots$.

2. The equations of motion of the massive Thirring model

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = \lambda/2\,\gamma^\mu\psi(\bar{\psi}\gamma^\mu\psi),$$

when $\psi_\alpha(x)$ are elements of a Grassmann algebra A with anticommuting generators $\psi_a(x)$, $\psi_a^*(x)$, $a = 1, 2$, and a is an involution in A, are also easier to consider in terms of the variables σ and τ. The conserved currents are calculated from the recurrence relations ($\psi_a^*\psi_a = \rho_a$)

$$b_{n+1} = \partial_\sigma b_n - i\lambda\rho_1 b_n - 2i\lambda\sum_{k+l=n,\ k\neq 0} b_k^*\psi_1 b_l + \psi_1\delta_{n+1,0};\quad b_n = 0,\ n < 0,$$

the dependence of b_n on τ is determined by the relations

$$\partial_\tau b_{n+1} = -m^2 b_n - i\lambda\rho_2 b_{n+1} + 2\lambda m\sum_{k+l=n} b_k^*\psi_2 b_l,\quad n \geqslant 0,$$

and the classical conservation laws take the form

$$\partial_\tau (\psi_1^* b_n - h.c.) = im\partial_\sigma(\psi_2^* b_{n-1} + h.c.).$$

Using the quantum equations of motion with effective Lagrangian

$$L = {}^i/_2(1+b)\bar{\psi}\overleftrightarrow{\partial}_\mu \gamma^\mu \psi - (m+a)\bar{\psi}\psi - \left[(\lambda + c)/_4\right](\bar{\psi}\gamma^\mu\psi)^2 ,$$

where a, b, and c are finite renormalizations of the mass, of the wave function, and of the charge, we obtain differential WI for the current

$$< N[\partial_\tau (\psi_1^* b_n - h.c.) - im/(1+b)\,\partial_\sigma(\{\psi_2^*\} b_{n-1} + h.c.)](x)\, X >$$

$$= \frac{m}{(1+b)^2} < N[m(\{\{\psi_1^*\}\} b_{n-1} - \psi_1^* \{\{b_{n-1}\}\} + \psi_1^* A_n) + 2a(\{\psi_1^*\} b_{n-1}$$

$$- \psi_1^* \{b_{n-1}\} + \psi_1^* B_n)](x) X>$$

$$+ \frac{(\lambda + c)m}{(1+b)^2} < N[\{\psi_2^* \rho_1\} b_{n-1} - \{\psi_2^*\}\rho_1 b_{n-1} + \psi_1^* C_n - h.c.](x)\, X> + \dots;$$

$$X = \prod_j \psi(x_j) \prod_i \bar{\psi}(y_i) .$$

The operators A_n, B_n, and C_n are calculated from the recurrence relations that follow from the relations for b_n and the equations of motion, and each curly bracket corresponds to one supersubtraction. The dots stand for the omitted Schwinger terms, which in the integral WI, after going to the mass shell, yield

$$(1 + c(\lambda, m))\Big(\sum_{j=1}^{m} (p_j^{\,\text{out}})^n - \sum_{j=m+1}^{l} (p_j^{\,\text{in}})^n\Big) < p_1, \dots p_m;\ \text{out}\ | p_{m+1}, \dots p_l;\ \text{in}>.$$

An analysis of the anomalous terms in the integral WI is carried out by using the Zimmermann identities [5]. However, in contrast to the sine-Gordon equation, the massive Thirring model is only a realizable theory and it is impossible to calculate explicitly the anomalous terms. Nonetheless, there are many simplifications connected with the fermion character of the field $\psi_\alpha(x)$, Lorentz invariance, and C and P invariance, which make it possible to analyze the anomalous terms fully and to show that they reduce, just as in the case of the sine-Gordon equation, to the corresponding Schwinger terms. Thus, the classical conservation laws are satisfied also in the quantum theory. The equivalence of the sine-Gordon equation and the massive Thirring model [7] makes it possible to state that there are no anomalies in the soliton sector of the $(\cos\phi - 1)_2$ quantum theory.

Just as this paper was sent to press, we learned of preprint [8] containing examples of conserved currents. Their expressions for $J_\mu^{(n)}$, $n = 3, 5, 7$ coincide with the currents obtained from the recurrence relations.

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