STABILITY OF CRITICAL STATES IN TYPE-II SUPERCONDUCTORS

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A method is proposed for calculating the stability limits of non-ideal type II superconductors against discontinuities. This method makes it possible to take into account differences in the thermal ambient, electromagnetic action of the shell, and other effects.

Placing a type-II superconductor in a magnetic field gives rise to characteristic instabilities in the form of flux jumps, which are well known to experimenters and can lead to premature transitions to the normal state. The instability was investigated theoretically, e.g., by Wipf [1] and by Swartz and Bean [2], who obtained stability criteria assuming the process to be fully adiabatic. A simplified solution [3], which to the contrary assumed full isothermy inside the superconductor, leads to the same criterion with a slight difference in the numerical coefficient. In the present note we propose a method for calculating the stability against jumps in the flux, wherein it is possible to take into account the heat exchange inside and outside the superconductor and a number of additional phenomena (e.g., the magnetocaloric effect, and, more interestingly, the electrodynamic interaction of the surrounding normal shell.

To derive the initial equation we make the usual assumption that the characteristic time of diffusion of the magnetic field in the resistive state is much shorter than the time of temperature diffusion. In other words, we assume that the current density always has time to become established at a level corresponding to the given temperature (we assume for the time being $\partial j_c/\partial H = 0$). Strictly speaking, our assumption is valid in this form only for sufficiently strong and only positive temperature fluctuations, but it is easy to see that the stability criterion obtained here provides a certain margin in the worst case.

We consider a plane superconducting layer of thickness 2b with initial current distribution (initial temperature $T = T_i$):

$$i = i_y = \begin{cases} -i_c(T_i) & 0 < x < b \\ i_c(T_i) & -b < x < 0\,. \end{cases} \tag{1}$$

The defining equation for the temperature is the usual heat-conduction equation, which we transform for small perturbations θ:

$$c\,\frac{\partial\theta}{\partial t} = \kappa\,\frac{\partial^2\theta}{\partial x^2} + i_y E_y(x) = \kappa\,\frac{\partial^2\theta}{\partial x^2} + i_c \int_0^x \mu_0\,\frac{\partial H_z}{\partial t}\,d\xi = \kappa\,\frac{\partial^2\theta}{\partial x^2} + \mu_0 i_c \int_0^x d\xi \int_x^b \frac{\partial i_y}{\partial t}\,d\eta\,. \tag{2}$$

When choosing the limits we took into account the fact that $E(0) \equiv 0$ in the first integral and $-\partial H(b)/\partial t \equiv 0$ in the second. We have put $B = \mu_0 H$, which is also quite customary in similar problems. To express all terms of the equation by means of a single variable we write, in accordance with our assumption, the connection between the current and the temperature in the form

$$i = i_c(T_i + \theta) = i_c(T_i) + \frac{\partial i_c}{\partial T}\,\theta = i_c - \frac{i_c}{T_0}\,\theta \tag{3}$$

Finally, we get rid of the double integral in (2) by differentiating (2) twice with respect to x:

$$c\,\frac{\partial}{\partial t}\,\frac{\partial^2}{\partial x^2}\,\theta = \kappa\,\frac{\partial^4\theta}{\partial x^4} - \frac{\mu_0 i_c^2}{T_0}\,\frac{\partial\theta}{\partial t}\,. \tag{4}$$

This equation describes the development of the perturbation in the linear approximation. We easily see that the only parameter determining the stability is the quantity

$$\beta = \frac{\mu_0\, i_c^2 b^2}{c\,T_0}\,. \tag{5}$$

Assuming furthermore

$$\theta = X\!\left(\frac{x}{b}\right) \exp\!\left(\lambda t\,\frac{\kappa}{c\,b^2}\right) \tag{6}$$

we obtain

$$X^{IV} - \lambda X^{II} - \lambda\beta X = 0 \, . \tag{7}$$

We have thus reduced the problem to a determination of the spectrum of the eigenvalues λ for Eq. (7) (unstable states correspond to $\lambda > 0$). Let us determine the necessary boundary conditions. Two conditions arise in connection with the transition from (2) to (4):

$$E(0) = 0 \text{ and } c\frac{\partial}{\partial t}\theta(0) = \kappa\frac{\partial^2}{\partial x^2}\theta(0) \text{ or } \lambda X(0) = X''(0) \tag{8}$$

$$\frac{\partial}{\partial t}H(b) \equiv 0 \text{ and } \frac{\partial E(b)}{\partial x} = 0 \quad \text{or} \quad \lambda X'(1) = X'''(1). \tag{9}$$

The remaining two conditions are determined by the character of the heat exchange on the boundaries of the layer. Let us assume, e.g., the adiabaticity condition (and also the condition of symmetry with respect to x)

$$X'(0) = X'(1) = 0 \, . \tag{10}$$

The characteristic equation for (7) is biquadratic, and the fundamental system (7) is expressed in terms of the elementary functions λ and β. Equating to zero the determinant corresponding to the system (8 - 10), we obtain the following equations for the determination of $\lambda(\beta)$:

$$k_1^3 \, \mathrm{th}\, k_1 = k_2^3 \, \mathrm{tg}\, k_2 \tag{11}$$

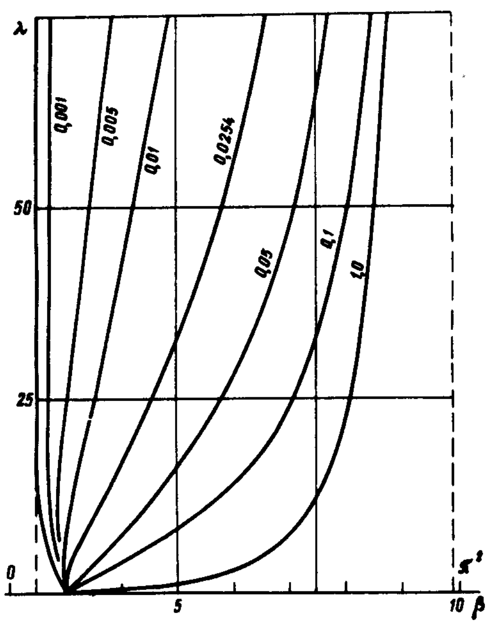

Fig. 1

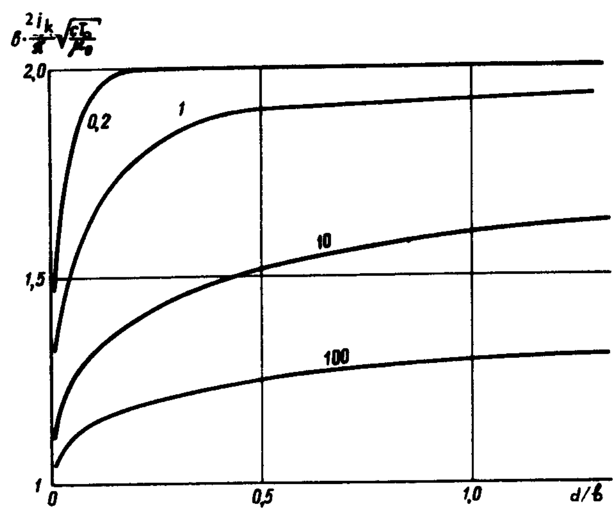

Fig. 2

Fig. 1. Plots of $\lambda(\beta)$ for the fundamental component of the perturbation at different values of d/b (marked on the curves) and $\alpha = 1$.

Fig. 2. Plot of the permissible superconductor thickness against the costing thickness for different α.

$$k_1 = \left(\sqrt{\lambda^2/4 + \lambda\beta} + \lambda/2\right)^{1/2}, \quad k_2 = \left(\sqrt{\lambda^2/4 + \lambda\beta} - \lambda/2\right)^{1/2}. \tag{12}$$

For the fundamental component (having no zeroes) of the perturbation, $\lambda(\beta)$ is represented by the extreme left curve on Fig. 1. The first $\lambda > 0$ occurs at $\beta = \pi^2/4$, corresponding precisely to the criteria of [1, 2]. As $\beta \to 3$ we have $\lambda \to +0$, which again agrees with the isothermal analysis [3] (the antisymmetrical component remains unstable also when $\beta > 3$).

Assume now that the superconductor is surrounded by a shell of normal metal of thickness d. Equation (7) and condition (8) obviously remain in force, and by virtue of (3) we can take θ to mean also the perturbation of the current in the superconductor. We let the conditions (10) also remain in force for the time being, without "using" the specific heat of the shell. For the current j_n in the normal metal we have the skin-effect equation

$$\frac{\partial i_n}{\partial t} = \frac{1}{\mu_0 \sigma_n} \frac{\partial^2 i_n}{\partial x^2} . \tag{13}$$

On the outer boundary we have H = 0, i.e., $\partial j_n/\partial x = 0$, and j_n can be written in the form

$$i_n = i_n(b + d)\exp\left(\lambda t \frac{\kappa}{c b^2}\right) \mathrm{ch} \sqrt{\frac{\lambda}{a}} \left(\frac{x - b - d}{b}\right). \tag{14}$$

The new parameter α is the ratio of the diffusion coefficient of the magnetic field in the normal metal to the temperature diffusion coefficient in the superconductor:

$$a = \frac{c_s}{\kappa_s \sigma_n \mu_0} . \tag{15}$$

In practice, α can be of the order of unity.

The two missing conditions express the continuity of E and H ($\partial E/\partial x$) on the boundary:

$$E_n = \frac{i_n}{\sigma_n} = E_s \quad \text{or} \quad \alpha\beta i_n(b) = X'(1) - \lambda X(1) \tag{16}$$

$$\alpha\beta b \frac{\partial}{\partial x} i_n(b) = X'''(1) - \lambda X'(1) \tag{17}$$

The determinant of the new system (8, 10, 16, 17) can also be represented in the form of the simple functions λ, β, α, and d. The function $\lambda(\beta)$ at $\alpha = 1$ and d = const is plotted in Fig. 1. The position of the point where $\lambda = 0$ remains the same for all d, i.e., the screening effect of the shell does not come into play as $\lambda \to 0$. However, even at $d \geq 8\alpha b/315$ it is not the fast ($\lambda \gg 1$) jumps but the slow ones ($\lambda \to 0$) which come into play first. At such thicknesses, therefore, it is necessary to take into account the specific heat of the shell (and in many problems also the transfer of the heat to the ambient). It is clear that the position of the point at which $\lambda \to 0$ is determined by the condition $\beta^* = 3$, and β^* (5) contains the total specific heat of all the components of the conductor. This point will thus shift to the right with increasing thickness of the coating (it becomes infinite in the presence of heat transfer). It turns out, however, that the upper branch of the $\lambda(\beta)$ curve passes to the left of the line $\beta = \pi^2$ as $\lambda \to \infty$ under all conditions. The corresponding perturbations describe flux jumps wherein the flux does not enter the conductor from the outside, but is only redistributed inside the conductor; in this case the shell has no effect. By way of example, Fig. 2 shows plots of the permissible superconductor thickness against the shell thickness for different α in the limiting case of ideal heat transfer ($X(1) = 0$).

It is clear that it is possible to include in this calculation scheme additional effects whose magnitudes are expressed in terms of the currents and temperature, and also to take into account the features of the heat transfer to the ambient, transfer inside the superconductor volume not occupied by currents, etc.

[1] S.L. Wipf, Phys. Rev. 161, 404 (1967).
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