Topological properties of a one-plaquette action and phase transitions in a lattice gauge theory

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The extrema of a one-plaquette action and their relationships with phase transitions on a space-time lattice are analyzed topologically. The results of numerical calculations of the topological susceptibility are discussed. The limit $N \to \infty$ is also discussed.

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Recent years have witnessed incontestable progress in our understanding of the structure of the strong interactions of elementary particles. The introduction of a lattice nonperturbative regularization has made it possible to use numerical methods to evaluate physical quantities directly from the quantum-chromodynamics Lagrangian.[1] Despite their real predictive value, however, numerical methods suffer from an obvious shortcoming in a lattice gauge theory: They ultimately generate a number for a physical quantity, and it is by no means always possible to follow the particular dynamics of the space-time lattice and the particular fluctuations of the gauge group with which this physical quantity is associated. We believe that far more information about a lattice gauge theory can be obtained by combining numerical and topological methods.

We consider the following space-time lattice of action:

$$S = \sum_p \sum_r \left(- \mathrm{Re}\, \beta_r \chi_r (U_p) \right), \tag{1}$$

where the summation is over the plaquettes of the lattice, the β_r are real numbers, and $\chi_r(U_p)$ is the trace of the matrix U_p in the representation with index r. In numerical calculations with, for example, the SU(2) group, the actions which are customarily used are the Wilson action $(\beta_{r=1/2} \neq 0, \beta_{r>1/2} = 0)$, the two-charge action $(\beta_{r=1/2} \neq 0, \beta_{r=1} \neq 0, \beta_{r>1} = 0)$, and the Manton action (with a suppressed large-r contribution). Action (1) is of course far more difficult to study by numerical methods, but it is the action which embodies information on the topological properties of the gauge fields on the lattice. With action (1) the lattice gauge theory acquires a rich phase structure; certain phases of the lattice gauge theory may be present in the continuous theory. Working from an analysis of the two-charge action, Bachas and Dashen[2] showed that the existence of a nondegenerate local extremum of a one-plaquette action is sufficient for the appearance of a phase transition in a four-dimensional lattice gauge theory. The end points of the first-order phase transition lie on a line on which the extrema of the one-plaquette action become unstable. In this letter we wish to show the topological origin of phase transitions in a lattice gauge theory with the one-plaquette action $S_p(\{\beta_r\}, U_p) = \Sigma_r(-\mathrm{Re}\beta_r \chi_r(U_p))$.

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The property $\chi_r(U_p) = \chi_r(gU_pg^{-1})$ for all $g \in G$ means that we can examine the function S_p on homogeneous spaces[3] (sets of orbits and sheets) with respect to inner automorphisms of the group. Homogeneous spaces of the group may have a complex topological structure. We consider the case of the gauge group SU(2). All the homogeneous spaces of the SU(2) group can be generated from the group unions of closed subgroups. The homogeneous spaces from the following[4]:

a) three-dimensional manifolds $SU(2) \approx S^3$;$SU(2)/C_n \approx L(n,1)$; $SU(2)/\widetilde{D}_{2n} \equiv \widetilde{L}_{2n}$;$SU(2)/T \equiv M_1$;$SU(2)|\widetilde{O} \equiv M_2$:$SU(2)/\widetilde{Y} \equiv M_3$;$n = 1,2, \ldots$. Here the symbol $\approx$ means a homeomorphism; $C_n, \widetilde{D}_{2n}, \widetilde{T}, \widetilde{O}, \widetilde{Y}$ are discrete subgroups; $L(n,1)$ is a lenslike space; and M_3 is the Poincaré space;

b) two-dimensional manifolds S^2; RP^2 is a real two-dimensional projective plane;

c) a zero-dimensional manifold homeomorphic to a point.

We might note that the fundamental group of homogeneous spaces with a discrete isotropic subgroup is isomorphic to the isotropic subgroup itself, so that the homogeneous spaces $L(n,1), \widetilde{L}_{2n}, M_1, M_2, M_3$ are topologically nonequivalent.

What is the orbit structure of the irreducible representations of the SU(2) group? It is not difficult to show that in the case of the semi-integer values $r = 1/2, 3/2, \ldots$ there are orbits S^3 and $L(p,1)$, $p = 3,5, \ldots, 2r$ $(r \neq 1/2)$; for the integer values $r = 1,2,3, \ldots$ the orbits are RP^2 if r is even or S^2 if r is odd. The orbits $\widetilde{L}_{2n}, M_1, M_2, M_3$ appear in the higher $(r \gtrsim 10)$ representations of the group.

We are interested in the behavior of the function $S_p(\{\beta_r\}, U_p)$ on homogeneous spaces and in its critical points.

We assume that the action $S_p(\{\beta_r\}, U_p)$ belongs to the class of polynomial functions on the group G; then it is a function $S_p(\{\gamma_r\}, k)$ of the polynomial invariants $k = \{k_\alpha\}, \alpha = 1, \ldots, q$ [for the SU(2) group we would have $q = 1$; for the SU(3) group we would have $q = 2$].[5] In this case the orbits are planes in the space of polynomial invariants, and we can write equations $F_l(k) = 0, l = 1, \ldots, s$, which determine the sheets, in terms of the polynomial invariants.[6] We introduce the Lagrange multipliers λ_l. An extremum on a sheet Ω is stable (with respect to small changes in $\{\gamma_r\}$) if for all points in the neighborhood of the extremal set $\{\bar{\gamma}_r\}$ the function $S_p(\{\lambda_r\}, k)$ has an extremum which varies continuously and differentiably on Ω.

If we define $\eta = (k,\lambda) = (k_1, \ldots, k_q; \lambda_1, \ldots, \lambda_s) = \{\eta_A\}_1 \leqslant A \leqslant q + s, \widetilde{S}_p(\{\gamma_r\}, \eta) = [S_p(\{\gamma_r\}, k) + \Sigma_{l=1}^{S} \lambda_l F_l(k)]|_{(k,\lambda) = \eta}$, then the equation $\partial_A \widetilde{S}_p(\{\gamma_r\}, \eta) = 0$ (2), $1 \leqslant A, B \leqslant q + s$, along with the condition $\det|\partial_A \partial_B \widetilde{S}_p(\{\gamma_r\}, \eta)|_{\eta = \bar{\eta}} = 0$ determines the stable extreme $\{\bar{k}_\alpha\}$ of the function $S_p(\{\gamma_r\}, k)$ on Ω which are in a mutually one-to-one correspondence with the solutions $(\{\bar{\gamma}_r\}, \bar{\eta})$ of Eq. (2).

It is also a straightforward matter to formulate the conditions for the instability of an extremum of $S_p(\{\gamma_r\}, k)$ (in the case of the two-charge action, they determine instability lines[2]). These instabilities may be of two types: (a) instabilities with respect to small changes in $\{\gamma_r\}$ within a single sheet; (b) instabilities with respect to a transition of an extremum from one sheet to another. The latter instabilities are particularly interesting for a lattice gauge theory, since there is a change in the topology of the manifold on which the function S_p is defined. Since we can introduce a measure and the concept of volume on homogeneous spaces, the search for such transitions can be carried out by Monte Carlo numerical methods.

Recent calculations of the topological susceptibility χ_t in a lattice gauge theory[7] have yielded a value two orders of magnitude lower than that which is required to solve the $U_A(1)$ problem of quantum chromodynamics. We believe that there are two reasons for this discrepancy: (1) the meager topology of the manifold on which the action is defined [usually on a subgroup of the icosahedron of the SU(2) group] and (2) the choice of the action itself, in which the contribution of higher representations of the gauge group is suppressed. The calculated value of χ_t in a lattice gauge theory apparently is unrelated to the continuum of the theory with lattice action (1), which is sensitive to the topology of the orbits and sheets.

For the groups $U(N)$ with $N\rightarrow\infty$ [the SU(N) and U(N) groups are indistinguishable in the limit $N\rightarrow\infty$] the orbits of maximum value are those which are constructed on the maximal torus $U^N(1)$ [for these orbits, the eigenvalues of the matrices $U\in U(N)$ are unequal: $U\in U(N)\lambda_1\neq\lambda_2\neq \ldots \neq\lambda_N$]. If $\lambda_i=\lambda_j$, the singular orbits have the isotropic subgroup $U(2)\otimes U^{N-2}(1)$. The effect of finite values of N (and the deviation from the semiclassical description, $N\rightarrow\infty$) stems from the increase in the volume of the orbits constructed on the discrete subgroups.

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