

Simplifying Experiments with Mandelbrot Set Model of Phase Transition Theory

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The study of Mandelbrot Sets (MS) is a promising new approach to the phase transition theory. We suggest two improvements which drastically simplify the construction of MS. They could be used to modify the existing computer programs so that they start building MS properly not only for the simplest families. This allows us to add one more parameter to the base function of MS and demonstrate that this is not enough to make the phase diagram connected.

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The problem of stability of time evolution is one of the most important in physics. Usually one can make the motion stable or unstable by changing some parameters which characterize Hamiltonian of the system. Stability regions can be represented on the phase diagram and transitions between them are described by catastrophe theory [1, 2]. It can seem that a physical system or a mechanism can be taken from one domain of stability to any other by continuous and quasi-static variation of these parameters, i.e. that the phase diagram is connected. However sometimes this expectation is wrong, because domains of stability can be separated by points where our system is getting totally destroyed. Unfortunately today it is too difficult to explore the full phase diagram for generic physical system with many parameters. Therefore, following [3], it was proposed in [4] to consider as a simpler model the discrete dynamics of one complex variable [5 – 8]. The phase diagram in this case is known as Universal Mandelbrot Set (UMS). MS is a well-known object in mathematics [9 – 11], but its theory is too formal and not well adjusted to the use in the phase transition theory. The goal of this paper is to make MS more practical for physical applications.

1. Structure of MS. First of all we remind the definition of MS and UMS from [4], which different from conventional definition in mathematical literature, see s.3 below. Mandelbrot Set (MS) is a set of points in the complex c plane. MS includes a point c if the map $x \to f(x,c)$ has stable periodic orbits. As shown in Fig.1 MS consists of many clusters connected by trails, which in turn consist of smaller clusters and so on. Each cluster is linear connected and can be divided into elementary domains where only one periodic orbit is stable. Different elementary domains can merge and even overlap. Boundary of elementary domain of n-th order, i.e. of a domain where an n-th order orbit is stable, is a real curve $c(\alpha)$ given by the system:

$$G_n(x,c) = 0, \quad F'_n(x,c) + 1 = e^{i\alpha}, \tag{1}$$

with

$$F_n(x,c) = f^{\circ n}(x,c) - x,$$

$$G_n(x,c) = \frac{F_n(x,c)}{\prod_m G_m(x,c)}, \text{ for all } n, \text{ divisors of } m.$$

Indeed, when $G_n(x,c)$ vanishes then x belongs to the orbit of exactly the n-th order. This orbit is stable if $|\frac{\partial}{\partial x} f^{\circ n}(x,c)| < 1$, what implies the second equation of (1). The solution of this system may give us more than a single n-th order domain. We say that $\mathrm{Resultant}(A(x),B(x)) = 0$ when functions $A(x)$ and $B(x)$ has common roots and $\mathrm{Discriminant}(A(x)) = 0$ when $A(x)$ has one or more multiple roots. Then domains of different orders merge at the points c where

$$\mathrm{Resultant}_x\Big(G_n(x,c), G_k(x,c)\Big) = 0. \tag{2}$$

Of course two orbits and thereafter domains merge only if n is divisor of k, so it is reasonable to consider only $\mathrm{Resultant}_x(G_n, G_{mn}) = 0$, with $m = 2, 3\ldots$ and $\mathrm{Discriminant}_x(G_n)$ for $k = n$.

Physically MS is a phase diagram of discrete dynamic of one complex variable. It is clear from Fig.1 that one should distinguish between three types of connectivity in different places of phase diagram. The first

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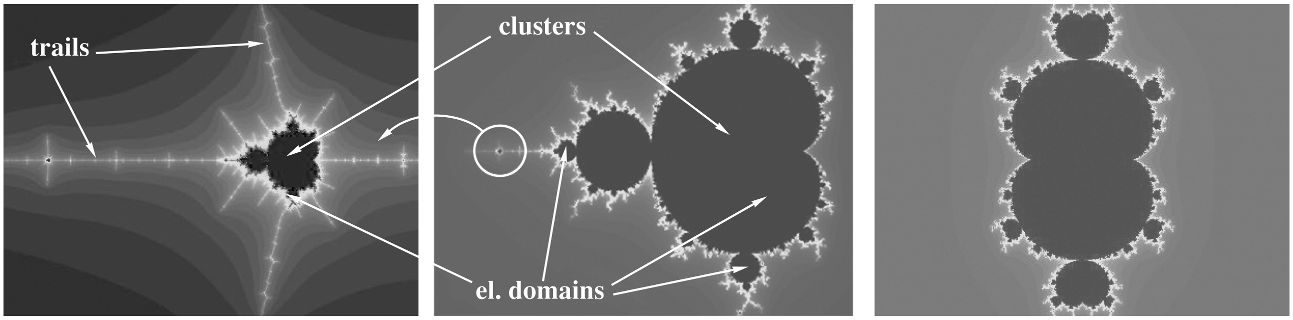

Fig.1. The simplest examples of Mandelbrot sets $MS(x^2+c)$ and $MS(x^3+c)$, constructed by Fractal Explorer [11]. The picture explains the terms"clusters","elementary domain" and "trails". It is difficult to see any clusters except for the central one in the main figure, therefore one of the smaller clusters is shown in a separate enlarged picture to the left

is linear connectivity: the possibility to connect any two points with a continuous line. The second type is weak connectivity: it means that only a closure of our set has linear connectivity. The third type we call strong connectivity: it means that any two interior points are connected with a thick tube.

Entire MS on Fig.1 is weakly[2)] but not linearly connected and its clusters are linearly, but not strongly connected. Universal Mandelbrot Set (UMS) is unification of MS of different 1_c-parametric families. When we rise from MS to UMS we add more parameters to the base function. Thus entire UMS could become strongly connected, but it is unclear whether this really happens.

2. Simplification of the resultant condition (2). In this section we prove that the resultant condition (2) can be substituted a by much simpler one:

$$\text{Resultant}_x\Big(G_n(x),(F_n'+1-e^{i\alpha})\Big)=0,\qquad \alpha=\frac{2\pi}{m}k. \tag{3}$$

To prove (3) it is enough to find the points where $\text{Resultant}_x(F_n,\frac{F_{nm}(x)}{F_n(x)})=0$. Then:

$$\text{Resultant}_x(F_n,\frac{F_{nm}(x)}{F_n(x)})=\text{Resultant}_x(F_n,\frac{F'_{nm}(x)}{F'_n(x)})=0.$$

By definition of $F(x)$:

$$F_{nm}(x)=F_{n(m-1)}\Big(F_n(x)+x\Big)+F_n(x).$$

Then:

$$\frac{F'_{nm}(x)}{F'_n(x)}=$$

$$=\frac{F'_{n(m-1)}(x)}{F'_n(x)}(F'_n(x)+1)+1\overset{(1)}{=}\frac{F'_{n(m-1)}(x)}{F'_n(x)}e^{i\alpha}+1=$$

$$=(\ldots((e^{i\alpha}+1)e^{i\alpha}+1)e^{i\alpha}+1)\ldots)+1=\sum_{l=0}^{m-1}e^{li\alpha}=\frac{e^{mi\alpha}-1}{e^{i\alpha}-1}.$$

Thus (2) implies that $e^{mi\alpha}-1=0$ and therefore $\alpha=2\pi k/m$.

This theorem is a generalization of a well known fact for the central cardioid domain of $MS(x^2+c)$ (see for example [9]). This also provides a convenient parametrization of generic MS.

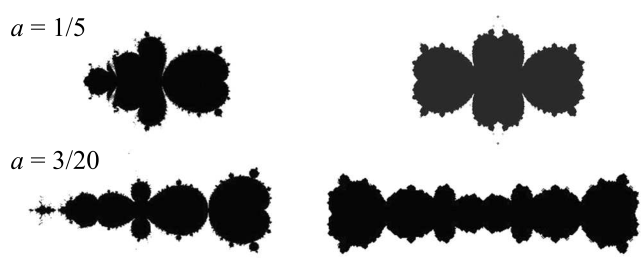

Fig.2. The families of Mandelbrot Sets $MS\left(a\cdot x^3+(1-a)\cdot x^2+c\right)$ at $a=1/5$ (the upper one) and $a=3/20$. The left pictures show the wrong Mandelbrot Set $\widetilde{MS}(x=0)$, generated by Fractal Explorer [11]. According to (4), the proper Mandelbrot Set is $MS=\widetilde{MS}(0)\bigcup\widetilde{MS}(-2(1-a)/3a)$, and it is shown in the right pictures. In this particular case, because of the Z_2 symmetry [4], the two different $\widetilde{MS}(x=0)$ are related by reflection w.r.t. axis $\text{Re}(c)=(a-1)(a+2)(2a+1)/27a^2$

3. A fast method of MS simulation. Historically MS was introduced in a different way from section 1. We call it $\widetilde{MS}$. It depends not only on the family of functions, but also on a point x_0. If c belongs to the $\widetilde{MS}(f,x_0)$ then

$$\lim_{n\to\infty}f^{\circ n}(x_0)\neq\infty.$$

In the literature one usually puts $x_0=0$ independently of the shape of $f(x)$. Such $\widetilde{MS}(f,0)\neq MS(f)$ except for the families like $f=x^a+c$. Existing computer programs [11] generate $\widetilde{MS}(f,0)$, and can not be used to

[2)] MS is usually claimed to be locally connected [9], i.e. any arbitrary small vicinity of a point of MS contains a piece of some cluster. In our opinion weak connectivity is another feature, especially important for physical applications.

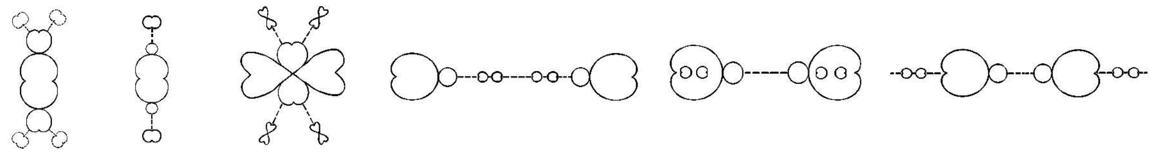

Fig.3. Schematic picture, showing the central cluster and the first non-trivial (order-2) clusters of $MS(a\cdot x^3+(1-a)\cdot x^2+c)$ at different values of a. The small clusters are artificially enlarged to make them visible. Their shapes imitate that of the central cluster, in accordance with [12]. All clusters move and pass through each other (i.e. the corresponding orbits are simultaneously stable at the same values of c, but remain different, the corresponding resultants do not vanish). This means that this 2_C-parametric section of UMS is still not linear connected

draw the proper $MS(f)$. Fortunately there is a simple relation:

$$MS(f)=\bigcup_{x_{cr}}\widetilde{MS}(f,x_{cr}), \qquad (4)$$

where union is over all critical points of $f(x)$, $f'(x_{cr})= = 0$. Equation (4) is closely related to hyperbolic and local connectivity conjectures [9]. It is also equivalent to the following two statements about the phase portrait in the complex x plane:

(I) If $\lim_{l\to\infty} f^{\circ l}(x_{cr})\neq\infty$ then there is a stable periodic orbit O of finite order which attracts x_{cr}. It implies that

$$MS(f)\supseteq\bigcup_{x_{cr}}\widetilde{MS}(f,x_{cr})$$

(II) If O is a stable periodic orbit, then a critical point x_{cr} exists, which is attracted to O. This implies that

$$MS(f)\subseteq\bigcup_{x_{cr}}\widetilde{MS}(f,x_{cr}).$$

The statement (I) says that if $\lim_{l\to\infty} f^{\circ l}(x_{cr})\neq\infty$ then this limit exists and is a stable orbit with finite period, i.e. that there are no such things as strange attractors in discrete dynamics of one complex variable. This statement is unproved but we have no counter-examples.

The statement (II) is much easier. If x_0 is a stable fixed point, then it is surrounded by a disk-like domain, where $|f'(x)|<1$. Its boundary is parametrized by $f'(x)=e^{i\alpha}$ and inside this area there is a point where $f'(x)=0$, i.e. some critical point x_{cr} of f. It is important that this entire surrounding of x_0 – and thus this x_{cr} – lie inside the attraction domain of x_0:

$$|f(x_{cr})-f(x)|<|x_{cr}-x|,$$

i.e. we found x_{cr} which is attracted to x_0. This argument can be easily extended to higher order orbits and can be used to prove (II).

Equation (4) leads to a simple upgrade of programs, which construct MS. Fig.2 demonstrates a result of this improvement.

4. A way from $MS(x^3+c)$ to $MS(x^2+c)$. As application of our results in ss.2 and 3 we consider the 2_C-parametric section of UMS for the family $f(x)=a\cdot x^3+(1-a)\cdot x^2+c$, which interpolates between $MS(x^3+c)$ at $a=1$ and $MS(x^2+c)$ at $a=0$. We extend consideration of [12] to non-trivial second order clusters which were beyond the reach of the methods used in that paper. The result is schematically shown on Fig.3. We used the Fast Method from section 3 to draw the entire MS. And we used simplification from section 2 to find the merging points of the first and the second domains and of the second and the forth domains. The important outcome of this experiment is that clusters, which were disconnected when $a=1$, remain disconnected for all a and go to the infinity when a goes to zero. There is no point between $a=0$ and $a=1$, where secondary clusters would touch the central cluster. Thus adding one more parameter to the MS does not make it linearly and strongly connected.

5. Conclusion. The Universal Mandelbrot Set is a representative model of sophisticated phase structure in complicated physical systems. However even this simpler model is still very difficult to explore and understand. In this paper we proposed considerable simplifications of the theory allowing to make the computer experiments with the properly defined MS in a simple and efficient fashion. This opens a way to attack the main puzzles such as the nature of trails and connectivity of phase diagram.

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