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From Simplified BLG Action to the First-Quantized M-Theory

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Concise summary of the recent progress in the search for the world-volume action for multiple M2 branes. After a recent discovery of simplified version of BLG action, which is based on the ordinary Lie-algebra structure, does not have coupling constants and extra dynamical fields, attention should be switched to the study of M2 brane dynamics. A viable brane analogue of Polyakov formalism and Belavin-Knizhnik theorem for strings can probably be provided by Palatini formalism for 3d (super)gravity.

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In the context of general string theory [1] a variety of string and superstring models, linked by a number of duality relations, is naturally unified in a hypothetical "M-theory" [2], which in its most naive perturbative phase is represented by "fundamental (super)membranes". The BLG action [3, 4] resolves the long-standing puzzle [5] of finding the 3d Lagrangian with appropriate superconformal symmetry and thus opens a way to developing the first-quantized theory of M2 branes. This implies that M-theory can now be studied in the same constructive manner as bosonic and super-strings in 1980's [6 – 8]. Naturally, this ground-breaking achievement attracts enormous attention [9 – 25], and some minor drawbacks of original analysis in [3, 4] are now fully cured.

The main obvious difficulty of original BLG formulation was its reliance upon sophisticated 3-algebra (quantum Nambu bracket) structure [26] – new for the fundamental physical considerations. The lack of experience and intuition about this structure caused certain confusion at the early stages: original BLG action was written only for an artificial(?) example of $SO(4)$ symmetry, problems were discovered with straightforward generalizations to other groups and even doubts appeared about the very existence of BLG action for the stack of N M2 branes with arbitrary N, which would require promotion of $SO(4)$ to $SU(N)$. The key step in overcoming this problem was analysis of the M2 $\to$ D2 conversion in [10], which linked the 3-algebra structure to conventional Lie algebras, governing Yang-Mills and D-brane theories. Based on this analysis, in [14] a "simplified" BLG action was introduced, which makes use of the Lie-algebra structure only (i.e. is based on "reducible-to-Lie-algebra" Nambu bracket of [27], see eq.(2) below). The only new ingredient, distinguishing this version of BLG action from the ones familiar from string/brane studies was a pair of extra color-less octuplets φ^I and χ^A. While very simple, this suggestion had serious problems as it was, originated from degeneracy of the underlying Nambu bracket and the lack of total antisymmetry of 3-algebra structure constants: this made original supersymmetry proof of [3] unapplicable and the action in [14] potentially non-supersymmetric. Thus it was meant to be a toy-example, showing the direction to eliminate unnecessary(?) elements of the BLG construction, but still possessing some extra fields and requiring some further tuning. A natural next step was to look at a central extension, lifting degeneracy of Nambu bracket [15] – and this was finally done in a triple of wonderful papers [25]. They resolved the discrepancy between [14] and BLG approach in an elegant way, by changing the nature of the extra fields φ, χ: they are actually auxiliary, non-dynamical variables. Kinetic terms $(\partial_\mu\varphi^I)^2$ and $\bar\chi^A\hat\partial\chi^A$ of [14] are substituted in [25] by $\partial_\mu\tilde\varphi^I\partial_\mu\varphi^I$ and $\frac{1}{2}\bar{\tilde\chi}^A\hat\partial\chi^A + \frac{1}{2}\bar\chi^A\hat\partial\tilde\chi^A$ where $\tilde\varphi, \tilde\chi$ is still another pair of color-less octuplets which do not appear anywhere else in the action and thus serve as Lagrange multipliers, eliminating the fluctuations of the unwanted φ and χ fields. In other words, the modified version of the simplified BLG action of [14] is now [25]:

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{tr}\Big(D_\mu\phi^I - B_\mu\varphi^I\Big)^2 + \frac{i}{2}\mathrm{tr}\psi^A\hat D\Big(\psi^A - \hat B\chi^A\Big) + \\
&+\Big(\partial_\mu\tilde\varphi^I - \mathrm{tr}(B_\mu\phi^I)\Big)\partial_\mu\varphi^I - \frac{i}{2}\bar{\tilde\chi}^A\hat\partial\chi^A - \\
&-\frac{i}{2}\bar\chi^A\Big(\hat\partial\tilde\chi^A - \mathrm{tr}(B_\mu\psi^A)\Big) + \frac{1}{2}\epsilon^{\mu\nu\lambda}\mathrm{tr}\Big(F_{\mu\nu}B_\lambda\Big) - \\
&-\frac{1}{12}\mathrm{tr}\Big(\varphi^I[\phi^J,\phi^K] + \varphi^j[\phi^K,\phi^I] + \varphi^K[\phi^I,\phi^J]\Big)^2 + \\
&+\frac{i}{2}\Gamma^{IJ}_{AB}\varphi^I\mathrm{tr}\Big(\bar\psi^A[\phi^J,\psi^B]\Big) + \frac{i}{4}\Gamma^{IJ}_{AB}\mathrm{tr}\Big(\bar\psi^A[\phi^I,\phi^J]\Big)\chi^B - \\
&-\frac{i}{4}\Gamma^{IJ}_{AB}\bar\chi^A\mathrm{tr}\Big([\phi^I,\phi^J]\psi^B\Big). \qquad (1)
\end{aligned}
$$

It essentially differs from eq.(21) of [14] in the second line[1)]. Here ϕ^I and ψ^A with $I = 1,\ldots,8$ and $A = 1,\ldots,8$ are real- and Grassmann-valued elements of the vector and spinor representations of the $SO(8)$ group respectively (related by octonionic triality to the second spinor representation, where the $N = 8$ SUSY transformation parameter takes values). They are also $N \times N$ matrices, i.e. belong to adjoint representation of the gauge group $G = SU(N)$. A_μ is the corresponding connection, also in the adjoint of G, $D_\mu\phi = \partial_\mu\phi - [A_\mu, \phi]$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and B_μ is an auxiliary adjoint vector field (not a connection). The other auxiliary fields $\varphi^I, \tilde\varphi^I$ and their superpartners $\chi^A, \tilde\chi^A$ are G-singlets (are color-less), possibly fragments of Kac-Moody extension of G. See [14, 25] for further details.

As explained in [25],

• The action (1) is $N = 8$ supersymmetric due to original BL theorem [3], because it is now based on the 3-algebra with totally antisymmetric structure constants, which is a central extension of the degenerate one [27] used in [14]:

$$[X,Y,Z] = \mathrm{tr}\,X\cdot[Y,Z] + \mathrm{tr}\,Y\cdot[Z,X] + \\ + \mathrm{tr}\,Z\cdot[X,Y] + \zeta\cdot\mathrm{tr}\,(X[Y,Z])\,, \qquad (2)$$

ζ is a central element, different from unity matrix I and related to it by non-trivial scalar product $<I,I> = <\zeta,\zeta> = 0$, $<I,\zeta> = <\zeta,I> = -1$, so that the 3-algebra metric is $\left(\begin{array}{cc|c} 0 & -1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & h \end{array}\right)$ and $f^{abcI} = -f^{abc}{}_\zeta = -f^{abc} = -f^{Iabc}$. The last term of (2) was absent in [14] and this made the 3-bracket degenerate and the structure constants (with the forth index raised by 3-algebra metric) not totally antisymmetric. The φ,χ fields are associated with the I (matrix-trace) generator, while $\tilde\varphi,\tilde\chi$ – with the central element ζ. Non-trivial 3-algebra metric implies the φ-$\tilde\varphi$ and χ-$\tilde\chi$ mixing form of the kinetic terms in (1).

• The would-be coupling constant in front of non-quadratic terms in (1) can be absorbed into rescaling of φ and χ, accompanied by rescaling of $\tilde\varphi$ and $\tilde\chi$ in the opposite direction. This lack of this feature was one of the problems in [14], and in (1) we have an action, which has no dimensionless coupling constants, as required in M-theory.

• All the unwanted extra fields $\varphi, \chi, \tilde\varphi, \tilde\chi$ are auxiliary: they do not propagate and contribute only through boundary terms[2)] (i.e. to correlators) and zero-modes.

• The fact that Lagrange multiplier $\tilde\varphi$ nullifies only $\partial^2\varphi$ rather than φ itself is very important, because this allows the zero-mode $\varphi = \mathrm{const}$. Among other effects, this zero mode can form a condensate, producing a term $<\varphi>^2\,\mathrm{tr}\,B^2$ from the first item in (1), which, after auxiliary field B_μ is integrated away, converts the Chern-Simons interaction $\mathrm{tr}F\wedge B$ into kinetic Yang-Mills term $<\varphi>^{-2}\,\mathrm{tr}\,F_{\mu\nu}^2$ for connection A_μ. This means that despite φ fields are now auxiliary, the crucially important possibility to use them for the M2 $\to$ D2 conversion *a la* [10] is preserved.

All this means that today we possess a perfectly simple version (1) of the BLG action for arbitrary number of M2 branes, there are no longer doubts about its existence for arbitrary gauge group $SU(N)$, there are no coupling constants, no extra dynamical fields, and it is clearly related to the other brane actions, as required by embeddings of $d = 10$ superstring models into the $d = 11$ M-theory. The road is now open for building up the first-quantized theory of M2 branes (supermembranes). This implies that attention can now be shifted from the study of 3-algebra structure (where a lot of interesting questions still remain) to the other issues: we know what should be the crucial next steps from the history of first-quantized theory of superstrings.

Constructing the action (1) can be considered as the very first step, corresponding to substitution of Nambu-Goto action for bosonic strings by a σ-model action, of which (1) is supposed to be a (super)membrane analogue. In the case of membranes the problem was more complicated, because Nambu-Goto action is ill (does not damp fluctuations) from the very beginning, no approach to bosonic membrane is still available (problems look more severe than the tachyon of bosonic string) and one should begin directly from the supersymmetric case, moreover supersymmetry should be immediately extended to $\mathcal{N} = 8$. Thus it may be not too surprising that we had to wait till 2008 to have this action written down...

In the case of strings the next big step was consideration of world sheets with non-trivial topologies, with two complementary formalisms finally developed for this purpose (and still not fully related, see [29] for description of the corresponding problems). One is the Polyakov formalism [6], promoting the σ-model action to arbitrary curved $2d$ geometries and general-

[1)] A trivial mistake of [14] in the ϕ^6 term (omitted item $2\sum_{I<J}\varphi^I\varphi^J\mathrm{tr}\Big([\phi^I,\phi^K][\phi_j,\phi^K]\Big)$ is also corrected in [25] and in (1).

[2)] In this respect the action in the φ-$\tilde\varphi$ sector is reminiscent of the one, considered in [28].

(Super)strings	(Super)membranes
$2d$ Nambu-Goto action $\rightarrow$ $2d$ σ-model action	spirit of membrane $\rightarrow$ simplified BLG action (1)
Polyakov formalism: introduction of $2d$ metric, critical dimensions (where massless excitations occur), sum over geometries, sum over topologies, relation to equilateral triangulations approach	BF-version of Palatini formalism in $3d$
Belavin-Knizhnik theorem: reduction of sum over metrics to sum over moduli	
topology of world sheet: spin structures and GSO projection, string field theory, boundary correlators and AdS/CFT correspondence	
topology of the target space (compactifications): generic $2d$ conformal theories, T-dualities, other dualities	
...	

izing the treatment of relativistic particle in [30]. Another is equilateral-triangulation approach, nicely expressed in terms of matrix models [31] and formally equivalent to substitution of smooth $2d$-geometries by Grothendieck's *dessins d'enfants* [32]. In the case of membranes this step is going to be a hard exercise, already because the topological classification of $3d$ world volumes is far more complicated than in $2d$. Still, the very first movement – introduction of $3d$ geometry into (1) by both above-mentioned methods – should be straightforward, and undoubtedly very interesting. For a variety of reasons it seems natural to do this in the modern BF-version of Palatini formalism, which is now widely popularized by controversial, but inspiring papers of G.Lisi [33]. Of certain help can be also comparison with the Green-Schwarz formalism for the superstrings [34], where world-sheet action has some common features with (1): it also looks non-linear, but actually non-linearities concern only the zero-modes and boundary effects.

Of crucial importance should be identification of the relevant world-sheet-geometry degrees of freedom (moduli), which the action is going to depend upon. This is not the $3d$ metric or dreibein and spin-connection themselves – already because of the general covariance. However, as we know from experience with strings, the remaining degrees of freedom (Liouville field) can also be irrelevant (or identified with the other physical fields [35]), so that the only remaining moduli are those of the $2d$ complex structures: finitely many for any given $2d$ topology. It is the analogue of Belavin-Knizhnik theorem [7, 8] that formulates this statement for strings, which should be the next big discovery in the story of BLG actions. Again, there are many complications in the case of membranes: as already mentioned, from the very beginning we need supersymmetry (and the corresponding problem for superstrings was partly resolved only quite recently! [36, 37]). Moreover, the analogue of Riemann theta-function theory [38] in $3d$ is not yet at our disposal – and here we should face the same problems as the other approaches to $3d$ topological theories [39]: there are no conventional terms to express our answers through...

In this short summary we do not speculate about the resolution of all these problems, i.e. about filling the empty spaces in the right column of the following Table. Our goal is to emphasize that we are now in front of the new and interesting breakthrough into the unknown – the possibility opened to us by the timely formulated problem [5], a brilliant insight [3, 4] and qualified polishing [9–24], culminated in [25] in the elegant formula (1), which is going to be – perhaps, in some reshaped and redecorated version – a new focus of attention in string theory in the coming years.

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