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Manifestation of Hamiltonian chaos in dissipative atomic transport in a standing-wave laser field

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We simulate atomic ballistic transport in a standing-wave laser field in the framework of a Monte Carlo stochastic wavefunction approach in which the coherent Hamiltonian evolution is interrupted at random times by spontaneous emission events. It is shown in numerical experiments and confirmed analytically that the character of spatial and momentum diffusion of spontaneously emitting atoms changes abruptly in the atom-laser detuning regime where the deterministic Hamiltonian dynamics has been shown to be chaotic. Thus, we find a manifestation of underlying Hamiltonian chaos in the diffusive-like center-of-mass motion which can be observed in real experiments.

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The mechanical action of light upon neutral atoms [1, 2] is at the heart of laser cooling, trapping, and Bose-Einstein condensation. Atoms in an optical lattice, formed by a laser standing wave, is an ideal system for studying quantum nonlinear dynamics. Operating at low temperatures and controlling the lattice parameters, experimentalists now are able to tailor practically one-dimensional potentials and manipulate with internal and external degrees of freedom of atoms. Experimental study of quantum chaos has been carried out with ultracold atoms interacting with a periodically modulated optical lattice [3]. To suppress spontaneous emission (SE) and provide a coherent quantum dynamics, atoms in those experiments were detuned far from the optical resonance. Adiabatic elimination of the excited state amplitude leads to an effective Hamiltonian for the translational (external) motion [4] of a 3/2 degree-of-freedom classical analogue which has a mixed phase space with regular islands embedded in a chaotic sea. De Brogile waves of ultracold atoms have been shown to demonstrate under appropriate conditions the effect of dynamical localization which means quantum suppression of chaotic diffusion [3, 4]. Decoherence due to spontaneous emission tends to suppress this quantum effect and restore classical-like dynamics [5].

A new arena of quantum nonlinear dynamics with atoms in optical lattices is opened if we work near the optical resonance and take the internal dynamics into account. In the Hamiltonian approximation, when one neglects SE, the coupling of internal and external atomic degrees of freedom has been shown to produce a number of nonlinear effects in rigid (i.e. without any modulation) optical lattices: chaotic Rabi oscillations, chaotic atomic transport, dynamical fractals, and Lévy flights [6, 7]. In reality the dynamics of atoms in near-resonant laser fields is not deterministic and continuous because of SE. The problem of interrelation between deterministic chaos and noise is rather general. Real systems are subject to noise which, usually, acts continuously. If the noise is comparatively weak we can study in which way it modifies the deterministic evolution solving perturbed Hamiltonian equations of motion. Spontaneous emission is a kind of a shot noise which is not small because it may cause the significant change in the internal state.

In this Letter we demonstrate analytically and numerically that the character of spatial and momentum diffusion of spontaneously emitting atoms changes qualitatively in the detuning regime where the underlying Hamiltonian dynamics has been shown to be chaotic in Ref. [6, 7].

In the frame rotating with the laser frequency ω_f, the Hamiltonian of a two-level atom in a strong standing-wave 1D laser field has the form

$$\hat{H} = \frac{\hat{P}^2}{2m_a} + \frac{1}{2}\hbar(\omega_a - \omega_f)\hat{\sigma}_z - \\ - \hbar\Omega\left(\hat{\sigma}_- + \hat{\sigma}_+\right)\cos k_f\hat{X} - i\hbar\frac{\Gamma}{2}\hat{\sigma}_+\hat{\sigma}_-, \qquad (1)$$

where $\hat{\sigma}_{\pm,z}$ are the Pauli operators for the internal atomic degrees of freedom, $\hat{X}$ and $\hat{P}$ are the atomic position and momentum operators, ω_a, ω_f, and Ω are the atomic transition, laser, and Rabi frequencies, respectively, and Γ is

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the spontaneous decay rate. Internal atomic states are described by the wavefunction $|\Psi(t)\rangle = a(t)|2\rangle + b(t)|1\rangle$ with a and b being the complex-valued probability amplitudes to find an atom in the excited $|2\rangle$ and ground $|1\rangle$ states. Note that the norm of the wavefunction, $|a|^2 + |b|^2$, is not conserved due to the non-Hermitian term in the Hamiltonian.

We use the standard Monte Carlo wavefunction technique [8] to simulate the atomic dynamics with the coupled internal and external degrees of freedom in an optical lattice. The evolution of an atomic state $|\Psi(t)\rangle$ consists of two parts: (1) evolution with continuously decaying norm of the atomic state vector without the emission of an observable photon and (2) jumps to the ground state ($a = 0$, $|b|^2 = 1$) each of which is accompanied by the emission of an observable photon at random time moments. The decaying norm of the state vector is equal to the probability of spontaneous emission of the next photon. It is convenient to introduce the new real-valued variables normalized all the time

$$u \equiv \frac{2\mathrm{Re}\,(ab^*)}{|a|^2 + |b|^2}, \quad v \equiv \frac{-2\mathrm{Im}\,(ab^*)}{|a|^2 + |b|^2}, \quad z \equiv \frac{|a|^2 - |b|^2}{|a|^2 + |b|^2}, \tag{2}$$

which have the meaning of synphase and quadrature components of the atomic dipole moment and the population inversion, respectively. Note that the length of the Bloch vector, $u^2 + v^2 + z^2 = 1$, is conserved.

Since we study manifestation of quantum nonlinear effects in a ballistic transport of atoms, when the average atomic momentum is very large as compared with the photon momentum $\hbar k_f$, the translational motion is described classically by Hamilton equations. The whole atomic dynamics is governed by the following Hamilton-Schrödinger equations [9, 10]

$$\begin{gathered}
\dot{x} = \omega_r p, \quad \dot{p} = -u \sin x + \sum_{j=1}^{\infty} \rho_j \delta(\tau - \tau_j), \\
\dot{u} = \Delta v + \frac{\gamma}{2} u z - u \sum_{j=1}^{\infty} \delta(\tau - \tau_j), \\
\dot{v} = -\Delta u + 2z \cos x + \frac{\gamma}{2} v z - v \sum_{j=1}^{\infty} \delta(\tau - \tau_j), \\
\dot{z} = -2v \cos x - \tfrac{\gamma}{2}(u^2 + v^2) - (z+1) \sum_{j=1}^{\infty} \delta(\tau - \tau_j),
\end{gathered} \tag{3}$$

where $x \equiv k_f \langle \hat{X} \rangle$ and $p \equiv \langle \hat{P} \rangle / \hbar k_f$ are normalized atomic center-of-mass (CM) position and momentum. The dot denotes differentiation with respect to the normalized time $\tau \equiv \Omega t$. The values of the normalized decay rate $\gamma \equiv \Gamma/\Omega$ and the recoil frequency $\omega_r \equiv \equiv \hbar k_f^2 / m_a \Omega$ are chosen to be $\gamma = 3.3 \cdot 10^{-3}$ and $\omega_r = 10^{-5}$, and they are similar to those used in experiments with cold Cs [11] and Rb atoms [12] in a strong laser field with the maximal Rabi frequency in the range 1–5 GHz. So, the normalized detuning between the field and atomic frequencies, $\Delta \equiv (\omega_f - \omega_a)/\Omega$, is a single variable parameter. In Eqs. (3) τ_j are random time moments of SE events and ρ_j are random recoil momenta whose projections on the axis x are between ± 1 (1D case). The rate of occurrence of SE events is $\gamma(z+1)/2$. The atomic variables change as follows just after jth SE: $p \to p + \rho_j$, $u \to 0$, $v \to 0$, $z \to -1$.

Equations (3) with $\gamma = 0$ and without the terms containing delta-functions describe Hamiltonian coherent evolution of the internal and external degrees of freedom of an atom without SE that has been shown [6] to be chaotic (in the sense of exponential sensitivity to small changes in initial conditions) in certain ranges of the parameters ω_r and Δ and initial momenta. In particular, it has been found that at $|\Delta| \lesssim 0.1$ atoms, whose energy $H \equiv \omega_r p^2/2 - u \cos x - \Delta z/2$ is in the range $0 < H < 1$, may wander in an optical lattice with alternating trapping in wells of the optical potential and flights over its hills. It is a specific kind of a random walking that may occur without any modulation of the lattice parameters and/or any noise like SE [6, 7].

In this Letter we study ballistic atoms which never change the direction of motion. It has been shown in Ref. [7] that without SE the momentum of a ballistic atom may oscillate in a deterministic but chaotic way around a mean value $\langle p \rangle$. In which way the chaos in the underlying Hamiltonian coherent evolution can manifest itself in the ballistic atomic transport that is inevitably stochastic process due to SE?

To answer the question we simulate Eqs. (3) by a Monte Carlo method (for details see [9]) and compute atomic trajectories in the momentum space to find the momentum diffusion coefficient D_p as a function of a current momentum p. To compare the behavior of D_p in the regimes with qualitatively different Hamiltonian CM dynamics we introduce a measure of Hamiltonian chaos $\Lambda \equiv \langle 2\theta(\lambda) - 1 \rangle$, where $\theta(\lambda)$ is a Heaviside function, which is equal to $\theta = 0$ for the maximal Lyapunov exponent $\lambda < 0$, $1/2$ for $\lambda = 0$, and 1 for $\lambda > 0$. The values of Λ in Fig.1 have been computed in the Hamiltonian limit with $\gamma = 0$) in Eqs. (3) by averaging over many atomic trajectories with close values of p_0. If $\lambda > 0$ with all those atoms, then $\Lambda = 1$, and we have Hamiltonian chaos with probability 1. If $\lambda = 0$ then $\Lambda = 0$, and the motion is regular with the probability 1. One gets $0 < \Lambda < 1$, if there are as chaotic as regular trajectories with close values of p_0. The chaos probability Λ is proportional to the fraction of atoms with positive

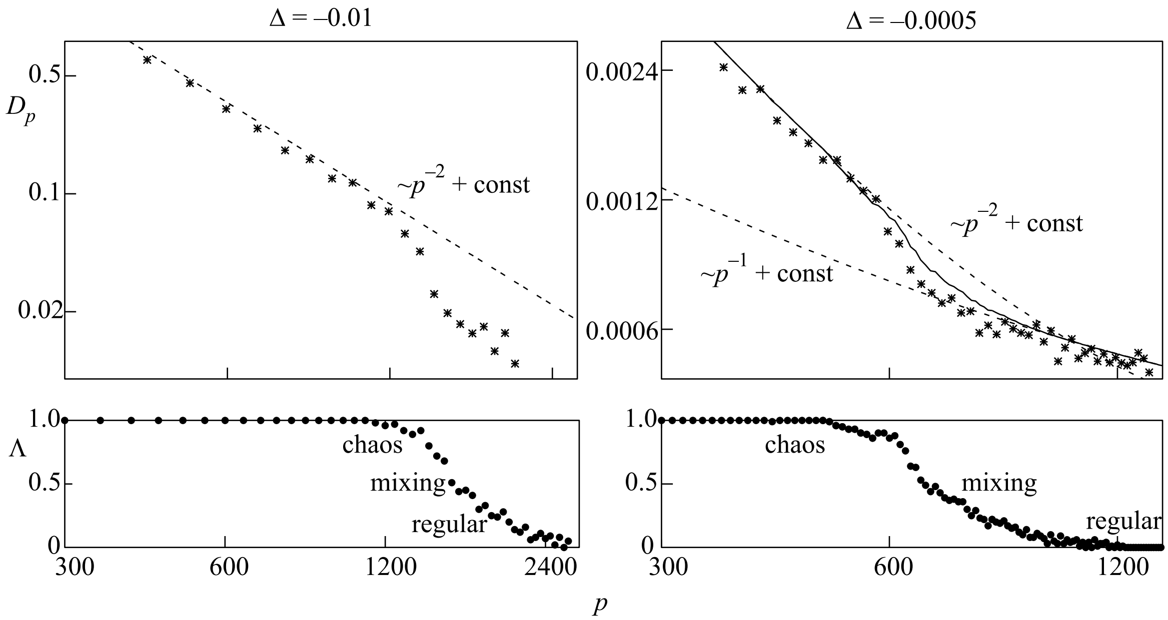

Fig.1. Correlation between the average momentum diffusion coefficient D_p in a log-log scale (in units of $\hbar^2 k_f^2 \Omega$) and probability of Hamiltonian chaos Λ in their dependencies on current atomic momentum p (in units of $\hbar k_f$) at $\Delta = -0.01$ and $\Delta = -0.0005$. Dashed lines with the slopes p^{-2} + const and p^{-1} + const are theoretical curves (8) and (11) valid in the regimes of Hamiltonian chaos and order, respectively. Solid line is a theoretical curve (12) derived to fit the exact numerical results. An abrupt change in the decay laws for D_p occurs for those values of p where the transition from order to chaos takes place in the underlying Hamiltonian dynamics

λs. Fig.1 demonstrates a prominent correlation between the regimes of chaotic (regular) Hamiltonian transport and the behavior of the momentum diffusion coefficient D_p of spontaneously emitting atoms. Beginning with those values of the momentum p, for which the probability of Hamiltonian chaos becomes smaller than 1 (see the lower panels in Fig.1), one observes an abrupt transition to a more regular regime of diffusive-like motion with another law of decay of D_p (see the upper panels in Fig.1).

Fig.1 illustrates a typical behavior of the function $D_p(p)$ with different values of the detuning at which Hamiltonian chaos may occur. For larger values $|\Delta| \gtrsim 0.1$ and smaller ones $|\Delta| \lesssim 0.00001$, the Hamiltonian dynamics becomes almost regular with all the initial momenta (see Fig.1 in [7]). We stress that the atomic transport in reality is stochastic with strong fluctuations of p due to SE even without Hamiltonian chaos. However, our simulations prove that the character of the momentum diffusion changes abruptly at those values of the current momentum where a transition from chaos to order occurs in the underlying Hamiltonian dynamics. This difference could be measured in real experiments and would provide us with direct signatures of atomic Hamiltonian chaos in terms of transport characteristics which are more easy to measure than the Rabi oscillations.

In what follows we will try to estimate the diffusion coefficient D_p analytically when the underlying Hamiltonian ballistic transport is chaotic and regular. In the weak Raman-Nath approximation, $\omega_r p^2/2 \gg |u \cos x + \Delta z/2|$, when the atomic kinetic energy is not strictly a constant, but much larger than the potential one, the momentum fluctuations between SE are small. In Ref. [10] we have shown that at small detunings, $|\Delta| \ll 1$, the total atomic energy H changes suddenly just after SE at $\tau = \tau_j$, and decreases linearly in between. We can treat the evolution of H as the following mapping:

$$H_j = H_{j-1} + \omega_r p(\tau_j^-)\rho_j + \frac{\omega_r}{2}\rho_j^2 + \frac{\Delta}{2} + \\ + u(\tau_j^-)\cos x(\tau_j) + \frac{\Delta}{2} z(\tau_j^-) + \frac{\Delta\gamma}{4}\langle 1 - z^2\rangle(\tau_j - \tau_{j-1}), \quad (4)$$

where H_j is a value of the energy just after jth SE, $u(\tau_j^-)$, $z(\tau_j^-)$, and $p(\tau_j^-)$ are values of the corresponding variables just before jth SE which are determined by coherent evolution at $\tau_{j-1} < \tau \le \tau_j^-$. The last term with the averaging over a time, exceeding the period of the Rabi oscillations, describes an energy drift between

SE events. In general, this random walk in the energetic space is asymmetric. The measure of momentum fluctuations is a momentum diffusion coefficient D_p that can be written in the weak Raman-Nath approximation as follows:

$$D_p \simeq \frac{\langle (H_j - H_{j-1})^2 \rangle - \langle H_j - H_{j-1} \rangle^2}{2\omega_r^2 p^2 \langle \tau_j - \tau_{j-1} \rangle}. \tag{5}$$

Using the largest terms in Eq. (4), we can estimate D_p to be

$$D_p \simeq \frac{\gamma}{12} + \frac{\langle u^2(\tau_j^-) \rangle \gamma}{8\omega_r^2 p^2}. \tag{6}$$

In deriving Eq. (6), we put $|\Delta| \ll 1$, $\langle u \cos x \rangle \simeq 0$, $\langle \tau_j - \tau_{j-1} \rangle \simeq 2/\gamma$, and $\langle \rho_j^2 \rangle = 1/3$.

To estimate the value of $\langle u^2(\tau_j^-) \rangle$ in Eq. (6), we note that at small detunings and in the absence of SE the variable u can be approximated by a constant while an atom moves between nodes of a standing wave (it fact, it performs shallow oscillations), and u changes suddenly when the atom crosses any node at $\cos x = 0$ [7]. Spontaneous emission results in additional jumps, $u \to 0$, but between SE events one can approximately describe the dynamics using the Hamiltonian theory. In the detuning range, where the underlying Hamiltonian motion is chaotic, the evolution of u can be approximated as a stochastic map

$$u_m \simeq |\Delta| \sqrt{\frac{\pi}{\omega_r p}} \sin \phi_m + u_{m-1}, \tag{7}$$

where u_m is a value of u after crossing the mth node between two SE events, ϕ_m are random phases in the range $[0, \pi]$. This map was derived from the expression (11) in Ref. [7] in the limit $|u| \ll 1$ which is justified because $u = 0$ after every act of SE, and u can never go far away from zero value because of small magnitudes of the jumps. The mean number of nodes an atom crosses between two successive SE events is $\langle M \rangle = 2\omega_r p/\gamma\pi$. Now we can estimate the value of $\langle u^2(\tau_j^-) \rangle \simeq \langle M \rangle \langle (u_m - u_{m-1})^2 \rangle \simeq \Delta^2/\gamma$ and, using Eq. (6), get the following formula for the momentum diffusion coefficient in the regime of Hamiltonian chaos:

$$D_p^{ch} \simeq \frac{\gamma}{12} + \frac{\Delta^2}{8\omega_r^2 p^2}. \tag{8}$$

In Fig.1 D_p^{ch} is shown by the dashed lines in a log-log scale which fit well numerical data in the range of atomic momenta where the underlying Hamiltonian dynamics is fully chaotic, i.e. at $\Lambda = 1$. The formula (8) is valid in a wide range of small detunings, but it does not work in the regime when the underlying Hamiltonian dynamics is mixed or regular.

The Hamiltonian motion is regular if $|\Delta| \ll 1$ and the atoms are sufficiently fast. Under such conditions, one may neglect momentum fluctuations between SE at all and adopt the simple linear law for the CM motion $x = \omega_r p \tau$ (the exact Raman-Nath approximation). In Ref. [7] we have derived a formula (see Eq. (A3) therein) for the value of u after crossing the first node. Using that formula, we have managed to get the map for u that is convenient to write down in the double-step form

$$u_m \approx 2\Delta \sqrt{\frac{\pi}{\omega_r p}}\, v_0 \cos\left(\frac{2}{\omega_r p} - \frac{\pi}{4}\right) + u_{m-2}, \tag{9}$$

where $v_0 = v(x = \pi n)$, $n = 0, 1, 2, \dots$. In the regular regime the jumps of u are not random, and the root-mean-square value is obtained using the map (9)

$$\langle u^2(\tau_j^-) \rangle \approx \frac{4\Delta^2 \omega_r p}{\gamma^2 \pi}\, v_0^2 \cos^2\left(\frac{2}{\omega_r p} - \frac{\pi}{4}\right), \tag{10}$$

where the average value of v_0^2 is estimated to be 1/2. Now we are ready to get from Eq. (6) the momentum diffusion coefficient in the regime of Hamiltonian regular motion

$$D_p^{\text{reg}} \simeq \frac{\gamma}{12} + \frac{\Delta^2}{8\gamma\omega_r p\pi}. \tag{11}$$

Thus, we have the analytic expressions for D_p in the regimes of Hamiltonian chaos ($\Lambda = 1$) and Hamiltonian order ($\Lambda = 0$). In the case of mixed motion, $0 \le \Lambda \le 1$, we suppose a linear law for the momentum diffusion:

$$D_p \simeq (1-\Lambda) D_p^{\text{reg}} + \Lambda D_p^{ch} \simeq \frac{\gamma}{12} + \frac{\Delta^2}{8\omega_r p}\left(\frac{1-\Lambda}{\gamma\pi} + \frac{\Lambda}{\omega_r p}\right). \tag{12}$$

This function, shown by the solid line in the upper right panel in Fig.1, fits rather well exact numerical results.

To conclude the Letter we propose a simple experimental scheme to confirm our main conjecture on an abrupt change in the character of atomic diffusion under conditions corresponding to two qualitatively different regimes of Hamiltonian CM motion, chaotic and regular ones. Let us consider a small atomic cloud moving in one direction with an average momentum $\langle p_c \rangle$. Initial position and momentum distributions are supposed to be a Gaussian with the standard deviations $\sigma_x^2 \equiv \langle (x - \langle x_c \rangle)^2 \rangle$ and $\sigma_p^2 \equiv \langle (p - \langle p_c \rangle)^2 \rangle$. The momentum diffusion coefficient is $D_p = d(\sigma_p^2)/(2d\tau)$. The temperature of the gas and its heating rate in Kelvins per second are

$$T \equiv \frac{2\langle E_k \rangle}{k_B} = \frac{\hbar^2 k_f^2 \sigma_p^2}{m_a k_B}, \quad \frac{dT}{dt} = \frac{2\hbar^2 k_f^2 \Omega D_p}{m_a k_B}, \tag{13}$$

where E_k is an atomic kinetic energy (in Joules) in the reference frame moving with the CM of the cloud. The heating rate is proportional to D_p which has been shown in this Letter to demonstrate different behavior in the regimes of regular and chaotic Hamiltonian dynamics.

The linear extent of the cloud is $L_X \equiv 2\sigma_x/k_f$. On a small observation time scale and with $\sigma_p \ll \langle p_c \rangle$, D_p is approximately the same for all the atoms in a cloud and could not change significantly during the observation time. Using the first equation in the set (3) and the expression for D_p written above, we obtain

$$\sigma_x^2 \approx \sigma_x^2(0) + \frac{1}{2}\omega_r^2\sigma_p^2(0)\tau^2 + \frac{2}{3}D_p\omega_r^2\tau^3. \qquad (14)$$

We have computed L_X with Eqs. (3) and with that formula with D_p given by (8) for a number of atomic clouds with different initial values of $\langle p_c \rangle$. In Fig.2 the dependence $L_X(\langle p_c \rangle)$ is plotted for $\Delta = -0.01$ at two moments of time. The analytic dashed lines fit well the exact numerical results in the range $\langle p_c \rangle \lesssim 1200$ where the underlying Hamiltonian dynamics is chaotic (see the left column in Fig.1). Note an abrupt change in the decay of $L_X(\langle p_c \rangle)$ beginning with those values of $\langle p_c \rangle \simeq 1200$ where the Hamiltonian motion becomes more regular.

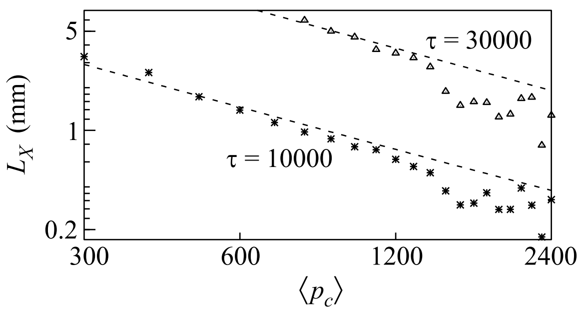

Fig.2. A log-log dependence of the cloud's linear extent L_X (in microns) on the average initial momentum $\langle p_c \rangle$ of atoms in a cloud at two moments of time. The analytic dashed lines were computed with the formula (14) with $D_p = D_p^{ch}$ valid in the chaotic regime on a short time scale. Note an abrupt change in the decay of $L_X(\langle p_c \rangle)$ in the range of $\langle p_c \rangle \simeq 1200$ where a chaos-order transition takes place in the underlying Hamiltonian motion. $\Delta = -0.01$, $2\sigma_x(0) = 0.5$, $2\sigma_p(0) = 5$, wavelength $\lambda_f = 2\pi/k_f = 850$ nm

In conclusion, we have found in numerical experiments the manifestation of dynamical instability and Hamiltonian chaos in ballistic motion of two-level atoms in a near-resonance standing-wave laser field. The effect of dynamical chaos in the fundamental atom-light interaction is masked by random events of SE. Nevertheless, we proved analytically and numerically that, under certain conditions, there exists a clear correlation between the behavior of the momentum diffusion coefficient D_p and Hamiltonian chaos probability Λ. To detect and quantify this effect in a real experiment, we propose to measure a linear extent L_X of atomic clouds with different values of the mean atomic momentum $\langle p_c \rangle$. We predict that beginning with those values of $\langle p_c \rangle$, for which Hamiltonian chaos probability becomes to be 1, the value of L_X for the corresponding atomic clouds should increase sharply.

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1. V. G. Minogin and V. S. Letokhov, *Laser Light Pressure on Atoms*, New York: Gordon and Breach, 1987.
2. A. P. Kazantsev, G. I. Surdutovich, and V. P. Yakovlev, *Mechanical Action of Light on Atoms*, Singapore: World Scientific, 1990.
3. F. L. Moore, J. C. Robinson, C. Bharucha et al., Phys. Rev. Lett. **73**, 2974 (1994); J. C. Robinson, C. Bharucha, F. L. Moore et al., Phys. Rev. Lett. **74**, 3963 (1995); S. Dyrting and G. J. Milburn, Phys. Rev. A **51**, 3136 (1995).
4. R. Graham, M. Schlautmann, and P. Zoller, Phys. Rev. A **45**, R19 (1992).
5. B. G. Klappauf, W. H. Oskay, D. A. Steck et al., Phys. Rev. Lett. **81**, 1203 (1998).
6. S. V. Prants and L. E. Kon'kov, JETP Letters **73**, 180 (2001) [Pis'ma v ZhETF, **73**, 200 (2001)]; S. V. Prants and V. Yu. Sirotkin, Phys. Rev. A **64**, 033412 (2001); S. V. Prants, JETP Letters **75**, 651 (2002) [Pis'ma v ZhETF **75**, 777 (2002)]; V. Yu. Argonov and S. V. Prants, JETP **96**, 832 (2003) [Zh. Eksp. Teor. Fiz. **123**, 946 (2003)].
7. V. Yu. Argonov and S. V. Prants, Phys. Rev. A **75**, 063428 (2007).
8. H. Carmichael, *An Open Systems Approach to Quantum Optics*, Springer-Verlag, Berlin, 1993; K. Molmer, Y. Castin, and J. Dalibard, J. Opt. Soc. Am. B **10**, 524 (1993).
9. V. Yu. Argonov and S. V. Prants, Acta Phys. Hung. B **26**, 121 (2006).
10. V. Yu. Argonov and S. V. Prants, Europhys. Lett. **81**, 24003 (2008).
11. H. Ammann, R. Gray, I. Shvarchuck et al., Phys. Rev. Lett. **80**, 4111 (1998); C. J. Hood, T. W. Lynn, A. C. Doherty et al., Science **287**, 1447 (2000).
12. W. K. Hensinger, N. R. Heckenberg, G. J. Milburn et al., J. Opt. B: Quantum Semiclass. Opt. **5**, R83 (2003).