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Bell's inequality with Dirac particles

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We study Bell's inequality using the Bell states constructed from four component Dirac spinors. Spin operator is related to the Pauli-Lubanski pseudo vector which is relativistic invariant operator. By using Lorentz transformation, in both Bell states and spin operator, we obtain an observer independent Bell's inequality, so that it is maximally violated as long as it is violated maximally in the rest frame.

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Relativistic entanglement and quantum nonlocality is investigated by many authors [1 – 28]. M. Czachor [7], investigated Einstein-Podolsky-Rosen (EPR) experiment with relativistic massive spin-$\frac{1}{2}$ particles. The degree of violation of the Bell's inequality is shown to depend on the velocity of the pair of particles with respect to the laboratory. He considered the spin singlet of two spin-$\frac{1}{2}$ massive particles moving in the same direction. He introduced the concept of a relativistic spin observable using the relativistic center-of-mass operator. For two observers in the lab frame measuring the spin component of each particle in the same direction, the expectation value of the joint spin measurement, i.e., the expectation value of the tensor product of the relativistic spin observable of each constituent particle, depends on the boost velocity. Only when the boost speed reaches that of light, or when the direction of the spin measurements is perpendicular to the boost direction, the results seem to agree with the EPR correlation. In the present article we use a relativistic spin operator for spin-$\frac{1}{2}$ particles which is constructed from Pauli-Lubanski pseudo vector. Also we use the four components Dirac spinors for constructing Bell states. With this spin operator the Bell's inequality is maximally violated and Bell observable does not depend on velocity of particles.

The Pauli-Lubanski pseudo vector is

$$W^{\mu} = \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}\sigma_{\nu\rho}\partial_{\sigma}, \tag{1}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_{\mu},\gamma_{\nu}]$ and in the Dirac representation the γ-matrices are

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix},\ \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},\ \gamma^5 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}. \tag{2}$$

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The invariant spin observable for Dirac particle is

$$\hat{s} = \frac{2W_{\mu}s^{\mu}}{m} = \pm\frac{1}{m}\gamma_5\, \not{s}\, \not{p} = \gamma_5\, \not{s}, \tag{3}$$

where s^{μ} is space-like normalized four-vector orthogonal to $\mathbf{p}$. The pervious equation holds for plane wave solutions, upper(lower) sign is related to positive(negative) energy solutions. For rest particles the vector s^{μ} becomes $(0,\mathbf{n})$, so that

$$\hat{s} = \gamma_5\gamma_0\mathbf{n}\cdot\gamma = \Sigma\cdot\mathbf{n}. \tag{4}$$

If we choose s along z axis, we see that following spinors are eigenstates of Σ_z

$$u(\mathbf{0},\frac{1}{2}) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ u(\mathbf{0},-\frac{1}{2}) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \tag{5}$$

$$v(\mathbf{0},\frac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},\ v(\mathbf{0},-\frac{1}{2}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{6}$$

with eigenvalues $+1$ for $u(\mathbf{0},\frac{1}{2})$ and $v(\mathbf{0},\frac{1}{2})$, and -1 for $u(\mathbf{0},-\frac{1}{2})$ and $v(\mathbf{0},-\frac{1}{2})$. Entangled Bell state corresponding to positive energy spinors $u(\mathbf{0},\pm\frac{1}{2})$ is defined as

$$\Phi_{11} = \frac{1}{\sqrt{2}}\left(u(\mathbf{0},\frac{1}{2})\otimes u(\mathbf{0},-\frac{1}{2}) - u(\mathbf{0},-\frac{1}{2})\otimes u(\mathbf{0},\frac{1}{2})\right). \tag{7}$$

The two particle states involve only positive energy spinors u and not the negative energy spinors v. This occurs because the positive energy states transformed among themselves separately and do not mix with each other under Lorentz transformation [2]. We consider measurements of the spins Σ_1 and Σ_2 along different directions performed on the correlated particles 1 and 2. For particle 1, we measure either $\hat{a}$ or $\hat{a}'$ where $\hat{a} = \mathbf{a}\cdot\Sigma_1$ and $\hat{a}' = \mathbf{a}'\cdot\Sigma_1$. Measured values of $\hat{a}$, $\hat{a}'$ are ± 1. Similarly, for particle 2, the quantity $\hat{b}$ or $\hat{b}'$ is measured, where $\hat{b} = \mathbf{b}\cdot\Sigma_2$ and $\hat{b}' = \mathbf{b}'\cdot\Sigma_2$. One of the form of Bell's inequality is

$$|\langle\hat{a}\otimes\hat{b}\rangle + \langle\hat{a}'\otimes\hat{b}\rangle + \langle\hat{a}\otimes\hat{b}'\rangle - \langle\hat{a}'\otimes\hat{b}'\rangle| \le 2. \quad (8)$$

This inequality is violated by the quantum mechanical results for, say, the singlet state (7) where $\langle\hat{a}\otimes\hat{b}\rangle = -\mathbf{a}\cdot\mathbf{b}$. It's easy to check that the maximally violation of Bell's inequality is equal to $2\sqrt{2}$.

To construct the spinors in an arbitrary Lorentz frame we boost the rest frame spinors by the standard boost from the rest frame to a frame with momentum $\mathbf{p}$. Then the entangled Bell states (7) takes the form

$$\Psi_{11} = \frac{1}{\sqrt{2}}\left(u\left(\mathbf{p},\frac{1}{2}\right)\otimes u\left(\mathbf{p},-\frac{1}{2}\right) - \right.$$
$$\left. - u\left(\mathbf{p},-\frac{1}{2}\right)\otimes u\left(\mathbf{p},\frac{1}{2}\right)\right). \quad (9)$$

Where the normalized positive energy spinors $u(\mathbf{p},\pm\frac{1}{2})$ are

$$u\left(\mathbf{p},\pm\frac{1}{2}\right) = \sqrt{\frac{E_p+m}{2E_p}}\begin{pmatrix}\varphi^{(\pm)}\\ \dfrac{\sigma\cdot\mathbf{p}}{E_p+m}\varphi^{(\pm)}\end{pmatrix}, \quad (10)$$

with $\sigma_3\varphi^{(\pm)} = \pm\varphi^{(\pm)}$. Now the Lorentz transformed spinors $u(\mathbf{p},\pm\frac{1}{2})$ are eigenstates of $\gamma_5\not{s}$ where s is the transform of s_z. In the frame in which the electron has momentum $\mathbf{p}$ the polarization vector is obtained by applying the Lorentz boost

$$s^\mu = (s_0,\mathbf{s}) = \left(\frac{\mathbf{p}\cdot\mathbf{n}}{m}, \mathbf{n} + \frac{(\mathbf{p}\cdot\mathbf{n})\mathbf{p}}{m(E_p+m)}\right). \quad (11)$$

Expectation value of the spin projection $\gamma_5\not{s}$ in state $u(\mathbf{p})$ is

$$\langle\gamma_5\not{s}\rangle = \frac{E_p+m}{2E_p}\times$$
$$\times\left(\varphi^\dagger(\sigma\cdot\mathbf{s})\varphi - \frac{\varphi^\dagger(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{s})(\sigma\cdot\mathbf{p})\varphi}{(E_p+m)^2}\right). \quad (12)$$

Using the following identity

$$(\sigma\cdot\mathbf{p})(\sigma\cdot\mathbf{s})(\sigma\cdot\mathbf{p}) = |\mathbf{p}|^2(s_z\sigma_z - s_y\sigma_y - s_x\sigma_x), \quad (13)$$

we have

$$u^\dagger\left(\mathbf{p},\frac{1}{2}\right)(\gamma_5\not{s})u\left(\mathbf{p},\frac{1}{2}\right) = \gamma^{-1}s_z, \quad (14)$$

$$u^\dagger\left(\mathbf{p},-\frac{1}{2}\right)(\gamma_5\not{s})u\left(\mathbf{p},-\frac{1}{2}\right) = -\gamma^{-1}s_z, \quad (15)$$

$$u^\dagger\left(\mathbf{p},\frac{1}{2}\right)(\gamma_5\not{s})u\left(\mathbf{p},-\frac{1}{2}\right) = s_x - is_y, \quad (16)$$

$$u^\dagger\left(\mathbf{p},-\frac{1}{2}\right)(\gamma_5\not{s})u\left(\mathbf{p},\frac{1}{2}\right) = s_x + is_y, \quad (17)$$

where $\gamma = E/m = (1-u^2)^{-1/2}$. Without loss of generality we assume that $\mathbf{p} = p\hat{z}$, then after some algebra the average of Bell operator on state Ψ_{11} to be

$$\langle\Psi_{11}|\hat{a}\otimes\hat{b}|\Psi_{11}\rangle = -\mathbf{a}\cdot\mathbf{b}, \quad (18)$$

which results the Lorentz invariant Bell's inequality. Here we compare our results with Czachor work [7]. He considered to spin singlet of two spin-$\frac{1}{2}$ massive particles moving in the same direction. The normalized operator corresponding to the spin projection along arbitrary direction $\mathbf{a}(\mathbf{a}^2 = 1)$ is [7]

$$\hat{a} = \frac{(\sqrt{1-\beta^2}\mathbf{a}_\perp + \mathbf{a}_\parallel).\sigma}{\sqrt{1-(\mathbf{a}\times\mathbf{u})^2}}, \quad (19)$$

where the subscripts $\perp$ and $||$ denote the components which are perpendicular and parallel to the boost speed $\mathbf{u}$. In this case the average of the relativistic EPR operator to be

$$\langle\Psi_{11}|\hat{a}\otimes\hat{b}|\Psi_{11}\rangle = -\frac{\mathbf{a}\cdot\mathbf{b} - u^2\mathbf{a}_\perp\cdot\mathbf{b}_\perp}{\sqrt{1-(\mathbf{a}\times\mathbf{u})^2}\sqrt{1-(\mathbf{b}\times\mathbf{u})^2}}, \quad (20)$$

Then against our results the expectation value is not Lorentz invariant and consequently the Bell's inequality is speed dependent.

In conclusion we have discussed EPR-type experiment with Dirac particles. For obtaining relativistic invariant Bell's inequality the spin operator for Dirac particles should be Lorentz transformed. Then in opposition of Czachor's results[7], since the degree of violation does not depend on momentums of particles, we have the relativistic invariant Bell's inequality.

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