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Non-conformal limit of AGT relation from the 1-point torus conformal block

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Given a $4d$ $\mathcal{N}=2$ SYM theory, one can construct the Seiberg-Witten prepotentional, which involves a sum over instantons. Integrals over instanton moduli spaces require regularisation. For UV-finite theories the AGT conjecture favours particular, Nekrasov's way of regularization. It implies that Nekrasov's partition function equals conformal blocks in $2d$ theories with W_{N_c} chiral algebra (virasoro algebra in our case). For $N_c=2$ and one adjoint multiplet it coincides with a torus 1-point Virasoro conformal block. We check the AGT relation between conformal dimension and adjoint multiplet's mass in this case and investigate the large mass limit of the conformal block, which corresponds to asymptotically free $4d$ $\mathcal{N}=2$ super symmetric Yang-Mills theory. Though technically more involved, the limit is the same as in the case of fundamental multiplets, and this provides one more non-trivial check of AGT conjecture.

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1. Introduction. $\mathcal{N}=2$ super symmetric Yang-Mills (SYM) theories have attracted attention for rather a long time, because they are ideally suited for the study of interplay between perturbative and non-perturbative effects and for manifestation of various dualities [1 – 4]. Depending on the fields content, these theories exhibit all types of renormalization behaviour of effective coupling constant g: it may tend to infinity (Landau pôle), and to zero (asymptotic freedom with dimensional transmutation in IR) or remain constant (UV-finite).

In $\mathcal{N}=2$ SYM theory the low-energy effective action is Abelian and its most important part is expressed in terms of the prepotential. Prepotential contains one-loop perturbative contribution and a far more sophisticated non-peturbative part, obtained as a sum over instantons. It was explicitly found by N.Seiberg and E.Witten (SW) [1, 2] with the help of duality arguments, and the answer was soon reformulated in terms of the spectral surfaces and simple integrable systems [5, 6]. The spectral curves were later interpreted in terms of branes. Straightforward evaluation of instanton sums is rather difficult, especially because some of the integrals over instanton moduli spaces diverge. See [7] for a comprehensive review and references.

A very successful direct caluculation was finally provided by N.Nekrasov [8]. He introduced a new partition function, depending on additional parameters ϵ_1 and ϵ_2, such that the limit $\epsilon_1,\epsilon_2 \longrightarrow 0$ reproduces SW prepotential.

Recently F. Alday, D.Gaiotto and Y. Tachikawa (AGT) made a ground-breaking conjecture that Nekrasov functions coincide with conformal blocks [9] of $2d$ Liouville/Toda models, and the ϵ-parameters are needed to allow arbitrary values of the central charge in their chiral W_{N_c} algebras (for $N_c=2$ the chiral algebra is just the ordinary Virasoro). AGT suggest a non-trivial association of conformal blocks with UV-finite $4d$ quiver models. The 4-point tree Virasoro block is associated with the $N_c=2$ gauge theory with $2N_c=4$ additional fundamental matter supermultiplets.

If there is instead, a single adjoint matter multiplet which also makes $4d$ theory UV-finite, the associated conformal block is the toric 1-point function. This claim was made in [10] and partly checked in [11]. We also confirm this relation and check it in one more way. Namely, we consider the limit of the large mass of adjoint mul-

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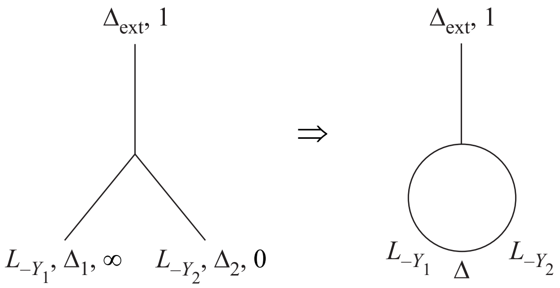

Triple vertex with two Virasoro descendants and the 1-point toric conformal block, obtained by taking a trace over Vermat module with a given dimension Δ. Each line is charaterized by dimension, by Ferrers diagram and external legs are also labeled by the position of the vertex operator on the Riemann surface

tiplet, where it decouples and the $4d$ theory turns into asymptotically free pure gauge $N = 2$ SYM. This pure gauge theory can be also obtained as the large-mass limit of the theory with 4 fundamentals, which implies that the corresponding limits of the tree 4-point and the toric 1-point conformal blocks (Figure) should be the same. The first limit has already been studied in [12, 13]. We find the second limit and show that it is indeed the same.

2. AGT relations. AGT hypothesis consists of several statements about relations between $2d$ CFT and $4d$ $\mathcal{N} = 2$ SYM theories. One of the statements is that perturbative part of Nekrasov partition function is equal to the product of DOZZ factors [14, 15], defining dependence of the triple functions in $2d$ Liouville theory on dimensions. Even more important and interesting is another part of this conjecture: the instanton part of Nekrasov partition function is equal to conformal block in $2d$ CFT (which depends on the chiral algebra, but not on the other details of $2d$ conformal model). Many examples were considered in [10] and later discussed in some detail [16 – 31, 11].

A list of many Nekrasov functions is available in numerous papers, starting from original [8]. More difficult is the situation with conformal blocks. Like Nekrasov functions they are formal series; in the simplest cases of interest in the present paper they are in one variable,

$$\mathcal{B}(x) = \sum_{n=0}^{\infty} x^n \mathcal{B}^{(n)}, \tag{1}$$

n is called the "level", and particular quantities $\mathcal{B}^{(n)}$ are built from two kinds of ingredients: Shapovalov form

$$Q_\Delta(Y_1, Y_2) = \frac{\left\langle L_{-Y_1} V_\Delta \middle| L_{-Y_2} V_\Delta \right\rangle}{\left\langle V_\Delta \middle| V_\Delta \right\rangle} \tag{2}$$

and two kinds of triple vertices [19]

$$\gamma_{123}(Y_1, Y_2, Y_3) = \frac{\langle L_{-Y_1} V_1(0) L_{-Y_2} V_2(1) L_{-Y_3} V_3(\infty)\rangle}{\langle V_1(0) V_2(1) V_3(\infty)\rangle}, \tag{3}$$

$$\bar{\gamma}_{12;3}(Y_1, Y_2, Y_3) = \frac{\left\langle L_{-Y_3} V_3 \middle| L_{-Y_1} V_1(1) L_{-Y_2} V_2(0) \right\rangle}{\left\langle V_3 \middle| V_1(1) V_2(0) \right\rangle}. \tag{4}$$

Here V are vertex operators, satisfying operator product expansions

$$V_1(x_1) V_2(x_2) = \sum_k (x_1 - x_2)^{\Delta_1 + \Delta_2 - \Delta_k} C_{12}^k V_k(x) \tag{5}$$

Operators are made from primaries by the action of Virasoro generators. Virasoro descendants are labeled by Young-Ferrers diagrams Y_i. Ferrers diagram is a sequence of integer numbers $k_1 \geq k_2 \geq k_3 \ldots$ So we define L_{-Y} as $L_{-Y} V = \ldots L_{-k_3} L_{-k_2} L_{-k_1} V$.

Using the integral definition of Virasoro operators one can get the following relation:

$$\begin{aligned} \left\langle L_{-n} V_1 \middle| V_2(1) V_3(0) \right\rangle &= \left\langle V_1 \middle| V_2(1) (L_n V_3)(0) \right\rangle + \\ &+ \left\langle V_1 \middle| (L_{-1} V_2)(1) V_3(0) \right\rangle + \\ &+ (1+n)\Delta_2 \left\langle V_1 \middle| V_2(1) V_3(0) \right\rangle + \\ &+ \sum_{k>0} C_{1+n}^{k+1} \left\langle V_1 \middle| (L_k V_2)(1) V_3(0) \right\rangle, \ \forall n. \end{aligned} \tag{6}$$

It is valid for arbitrary fields V_i, not obligatory primary ones [19]. Using this formula we can calculate all needed $\bar{\gamma}_{12;3}$. The 4-point conformal block was computed already by many authors, because in this case we need only $\bar{\gamma}_{12;3}(\emptyset, \emptyset, Y)$, for which there is the well known general formula. The 1-point torus conformal block, which is of interest for us here, is made from a more complicated $\bar{\gamma}_{12;3}(\emptyset, Y_2, Y_3)$, which is not yet known in the general form. Thus we need to compute these vertices one by one.

Writing the correlator of 4 fields and expanding it with the help of (5) and using recently introduced notations we get

$$\mathcal{B}^{(n)}_{4-\text{point}} = \sum_{|Y_\alpha| = |Y_\beta| = n} \bar{\gamma}_{12;\alpha}(Y_\alpha) Q_\Delta^{-1}(Y_\alpha, Y_\beta) \gamma_{\beta 34}(Y_\beta). \tag{7}$$

It is clear that to compute the conformal block one should use γ instead of each vertex and Q^{-1} instead of inner lines.

From the AGT relation for the 4-point conformal block we obtain

$$\Delta = \frac{\epsilon^2 - 4a^2}{4\epsilon_1\epsilon_2},\ c = 1 + \frac{6\epsilon^2}{\epsilon_1\epsilon_2}. \qquad (8)$$

In these relations a is a v.e.v. of the $4d$ SYM theory. They were originally obtained in [10] and [18]. Nevertheless we defined one external dimension and power of η-multiplier.

3. 1-point conformal block on a torus. The formula to calculate 1-point torus conformal block (Figure) is

$$\mathcal{B}(q) = \sum_n q^n \mathcal{B}^{(n)} = \qquad (9)$$

$$= \sum_{Y_1,Y_3} q^{|Y_1|} \left\langle L_{-Y_1}V_1 \middle| V_{ext}(1) L_{-Y_2}V_2(0)\right\rangle Q_\Delta^{-1}(Y_1,Y_2).$$

Besides x it depends on two dimensions, Δ and Δ_{ext} and on the central charge c. AGT conjecture identifies this conformal block with analogous expansion of Nekrasov partition function

$$\widetilde{\mathcal{N}}(q) = \left(q^{-1/24}\eta(q)\right)^\nu \sum_{n=0}^{\infty} q^n \mathcal{N}^{(n)} = \sum_{n=0}^{\infty} \widetilde{\mathcal{N}}^{(n)}, \quad (10)$$

$$\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$$

is the Dedekind eta function, $q = e^{2\pi i\tau}$, $\tau = 4\pi i/g^2 + \theta/2\pi$ is complex coupling constant. $\mathcal{N}^{(n)}$ depends on the v.e.v. modulus a, on adjoint multiplet's mass m and also on ϵ_1 and ϵ_2.

3.1. *The First Level.* The AGT relation $\{\Delta, \Delta_{ext}, c\} \overset{\nu}{\leftrightarrow} \{a, m, \epsilon_1, \epsilon_2\}$ can be found from equality $\mathcal{B}^{(1)} = \widetilde{\mathcal{N}}^{(1)}$ at level one. Explicitly

$$\mathcal{B}^{(1)} = \frac{\Delta_{\rm ext}^2}{2\Delta} - \frac{\Delta_{\rm ext}}{2\Delta} + 1, \qquad (11)$$

while

$$\mathcal{N}^{(1)} = \frac{(\epsilon_1 - m)(\epsilon_2 - m)}{\epsilon_1\epsilon_2(\epsilon^2 - 4a^2)}(-8a^2 + 2\epsilon^2 - 2\epsilon m + 2m^2). \qquad (12)$$

These quantities coincide provided (1) is suplemented by

$$\Delta_{\rm ext} = \frac{m(\epsilon - m)}{\epsilon_1\epsilon_2},\ \nu = 1 + \frac{2m(m-\epsilon)}{\epsilon_1\epsilon_2}. \qquad (13)$$

The answer was computed with the help of *ad hoc* triple conformal correlator with a non primary field [34]. As we already noticed this computation is non trivial because it involves the vertex $\bar{\gamma}_{23;1}$ with two non-trivial Young diagrams, see [34] for details.

3.2. *The Second Level.* The first non-trivial check of AGT conjecture is at level two. We made this check and there is indeed a complete coincidence between conformal block and Nekrasov partition function holds at level two, as already claimed in [11]. Unfortunately, the full formula is too cumbersome to be presented here.

Instead in this paper we concentrate on additional check, which can be extended to all levels: we investigate the limit of large m. According to (13) this is the same as large $\Delta_{\rm ext}$, and what we need is a new asymptotics: (15) Together with (14) this gives an insight: only particular terms dominate in the limit.

4. Large Mass Behaviour. AGT relation is originally formulated for UV-finite gauge theories in $4d$. Asymptotically free pure gauge theory arises when masses of additional matter supermultiplets are led to infinity, while the bare coupling $x \sim q_0$ is simultaneously led to zero. In the case of adjoint multiplet the product $xm^2 = \Lambda^4$ is kept constant in this scaling limit. We know that if we have a large mass, we also have large $\Delta_{\rm ext}$.

With the aid of (13) one can obtain asymptotic behaviour of the first and the second order term of the conformal block

$$\mathcal{B}^{(1)} \underset{m\longrightarrow\infty}{\asymp} \frac{\Delta_{\rm ext}^2}{2\Delta} = \Delta_{\rm ext}^2 Q_\Delta^{-1}([1],[1]). \qquad (14)$$

$$\mathcal{B}^{(2)} \underset{m\longrightarrow\infty}{\asymp} \Delta_{\rm ext}^4 Q_\Delta^{-1}\left([1^2],[1^2]\right). \qquad (15)$$

Substituting Λ in (14), (15) and (1) and generalizing this formula we can guess that the large mass limit of $\mathcal{B}$ looks like

$$\lim_{\substack{m\longrightarrow\infty \\ xm^2=\Lambda^4={\rm const}}} \mathcal{B}(x) = \sum_{n=0}^{\infty} \Lambda^{4n} Q_\Delta^{-1}([1^n],[1^n]). \qquad (16)$$

We also have an explanation why this formula is correct. When one is studying high mass or in other notations large $\Delta_{\rm ext}$ limit, one should focuse only at the term with the highest power of $\Delta_{\rm ext}$, because of (13).

First of all we should describe how we evaluate $\mathcal{B}^{(n)}$:

$$\mathcal{B}^{(n)} = \sum_{|Y_i|=|Y_j|=n} \left\langle L_{-Y_i}V_1 \middle| V_2(1)L_{-Y_j}V_3(0)\right\rangle Q_\Delta^{-1}(Y_i,Y_j). \qquad (17)$$

One can see that Q depends only on Δ, so all $\Delta_{\rm ext}$ dependence is concentrated in γ. We will prove a theorem that the highest power of $\Delta_{\rm ext}$ in

$$\left\langle L_{-Y_i}V_1 \middle| V_2(1)L_{-Y_j}V_3(0)\right\rangle \qquad (18)$$

is equal to the total number of Virasoro operators in Y_i and Y_j. If this theorem is correct then the highest power of Δ_{ext} is in the term where $Y_i = Y_j = [1^n]$. We also will prove that the coefficient in front of the highest power of Δ_{ext} in the scalar product for this term is equal to one. Therefore the leading term in $\mathcal{B}^{(n)}$ when Δ_{ext} is large indeed looks like: $\Delta^{2n} Q_\Delta^{-1}([1^n],[1^n])$.

5. Proof of the theorem. In this section we prove that the highest order Δ_{ext} in scalar product of three fields (18) is equal to the total number of Virasoro operators in this scalar product. Also the coefficient in front of this highest order term will be found.

Using relation (6) with $n = 0$ one can show that

$$\left\langle V_1 \middle| L_{-1} V_2(1) V_3(0) \right\rangle = \widetilde{\Delta}_{\text{ext}} \left\langle V_1 \middle| V_2(1) V_3(0) \right\rangle. \quad (19)$$

Here and below we use the notation $\widetilde{\Delta}_{\text{ext}} = \Delta_{\text{ext}} + (\text{arbitrary function of } \Delta)$ and we can use $\widetilde{\Delta}_{\text{ext}}$ instead of Δ_{ext} because we are only interested in the highest power of the Δ_{ext} term.

We can use the induction method to prove our statement. So the base of induction is the zero level – if there are no Virasoro operators then γ does not depend on Δ_{ext} because it is equal to one. Using (19) we can see that two terms in the second row of (6) can be summed directly and give us $n\widetilde{\Delta}_{\text{ext}}$ along with lowering the number of the Virasoro operators in front of V_1 by one. From the Virasoro commutation relation we know that $L_k L_{-n} V_3 = L_{-n} L_k V_3 + (n+k) L_{k-n} V_3 + \delta_{k,n} \frac{cm(m^2-1)}{12}$. So if V_3 is a primary field, as it is in our case, $L_k L_{-Y} V_3$ is the sum of the fields with the same number of operators as $L_{-Y} V_3$, and the coefficients of this sum depends only on Δ. So the order of this term is lower than the order of the second row in (6). The third row in our case is equal to zero, because in our case V_2 is always a primary field.

So using previous statements one can see that reducing the number of the operators in front of V_1 gives us multiplication by $n\widetilde{\Delta}_{\text{ext}}$ from the second row plus some terms of the lower order, where n is the order of the Virasoro operator which we expanded. And similarly reduction of the number of operators in front of V_3 also give the $n\widetilde{\Delta}_{\text{ext}}$ multiplier.

Therefore each time when one reduces the total number of operators in (18) by one he gains a product of $\widetilde{\Delta}_{\text{ext}}$, the function of Δ and the expression of similar form but lower order. Also one can see that the coefficient in front of the highest power of Δ_{ext} in (18) is equal to

$$\prod_{L_{-p} \in Y_i} p \prod_{L_{-q} \in Y_j} q. \quad (20)$$

So we not only proved that the highest power is equal to the total number of Virasoro operators but also counted the coefficient in front of this term. And indeed this coefficient is equal to one if $Y_i = Y_j = [1^n]$.

6. Conclusion. We studied the large-mass limit of the AGT relation for the $N = 2$ SYM theory with $N_c = 2$ and adjoint matter multiplet. The corresponding limit of the 1-point conformal block on a torus reproduces the answer, obtained earlier by taking the similar limit of the 4-point tree conformal block, associated through AGT relation with the $4d$ theory with $N_f = 2, N_c = 4$ fundamental multiplets. The fact that the two limits coincide is implied by AGT relation, but is somewhat non-trivial from the point of view of CFT. Identity involves two very different conformal blocks, and even the relevant triple vertices are different in two cases: in toric case more sophisticated vertices with two Virasoro descendants are needed. Thus we obtained one more non-trivial confirmation of the AGT conjecture.

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