Pis'ma v ZhETF, vol. 90, iss. 11, pp. 826 – 829

Oscillator heating by the colored noise

*P. V. Elyutin, N. S. Maslova, N. A. Gippius**+

Moscow State University, Department. of Physics, 119991 Moscow, Russia

**LASMEA, UMR 6602 CNRS, Université Blaise Pascal, 63177 Aubière, France*

+*A.M. Prokhorov General Physics Institute RAS, 119991 Moscow, Russia*

Submitted 31 August 2009
Resubmitted 28 October 2009

The problem of energy kinetics of the harmonic oscillator under the influence of the colored noise is studied in a novel approach that describes the evolution by a discrete time random walk with randomly varying step. In this approach the variations of the oscillator's energy on adjacent time intervals happen to be virtually uncorrelated even for large correlation times of the noise. The average time of the first passage of the oscillator with the initial zero energy across some threshold value is calculated. The pre-exponent factor of transition rate is found to depend on the parameters of noise and not on oscillator damping and correctly describes the case of zero friction. The agreement in exponential factors obtained by the suggested approach and kinetic equation is demonstrated for narrow-band colored noise.

PACS: 02.50.Fz

For dynamical system with damping influenced by colored noise the analysis of the average time of the first passage through the particular energy state is of great interest. In order to understand the main features of fluctuation induced transitions for such systems the simplest models (e.g. harmonic oscillator) have to be investigated. Some progress has been achieved in the investigation of the exponential factor of the transition rate [1, 2]. However, the dependence of the pre-exponential factor on the system parameters and noise characteristics is much less understood.

We study the evolution of the harmonic oscillator,

$$m\left(\ddot{x}+\gamma\dot{x}+\omega^2 x\right)=F\xi\left(t\right), \tag{1}$$

where m is the particle mass, γ is the damping coefficient, and ω is the oscillator's eigenfrequency, under the influence of the colored noise $F\xi\left(t\right)$. The random process $\xi(t)$ is assumed to be the normally distributed (gaussian) process with the exponential correlation function,

$$\langle\xi\left(t\right)\rangle=0, \quad \langle\xi\left(t\right)\xi\left(t+\tau\right)\rangle=e^{-\nu|\tau|}, \tag{2}$$

where ν is the rate of relaxation of the noise correlations. The force amplitude F is supposed to be sufficiently small. The term "energy" denotes the full energy of the unperturbed oscillator, that is $E\left(t\right)=$ $=m\dot{x}^2/2+m\omega^2x^2/2$.

We intend to calculate the average time of the first passage of the oscillator (with the initial zero value of the energy) across the energy threshold E^*. This problem is a part of the description of the escape of the system (with one degree of freedom) from the metastable state under the influence of the external noise. The paradigm of the theory of this process was established by Kramers [3] through the expression for the rate of the escape across the energy barrier of the height ΔE:

$$\Gamma=C\exp\left(-\frac{\Delta E}{kT}\right), \tag{3}$$

where the pre-exponential factor C has the dimensionality of the inverse time. The temperature in energy units kT is introduced through the autocorrelator of the perturbing force with white noise properties, $\langle\xi\left(t\right)\xi\left(t+\tau\right)\rangle=2m\gamma kT\delta\left(\tau\right)$. The problem of the first passage time was recently treated by Margolin and Barkai [4]. They have obtained the asymptotic forms of the distribution of times, but have not reached the expression for the average time. Furthermore, these authors have restricted their study by the case of white noise.

If the unperturbed motion of the oscillator is a harmonic motion with the amplitude A, $x\left(t\right)=A\sin\omega t$, then the energy variation during one period of the unperturbed motion $T=2\pi/\omega$ in the first order of the perturbation theory is

$$\Delta E=F\omega\int\limits_0^{2\pi/\omega}A\cos\omega t\,\xi\left(t\right)dt. \tag{4}$$

The autocorrelation function of variations of energy in two motion periods with centers, separated by k peri-

ods, is given by $B_{\Delta E}(k) = \langle \Delta E_n \Delta E_{n+k} \rangle = F^2\omega^2 A^2 J_k$, where the integrals J_k are

$$J_k = \int_0^{2\pi/\omega} dt_1 \int_{-k(2\pi/\omega)}^{-(k-1)(2\pi/\omega)} dt_2 \Phi(t_1, t_2), \qquad (5)$$

$$\Phi(t_1, t_2) = \cos\omega t_1 \ \cos\omega t_2 e^{-\nu|t_1 - t_2|}.$$

The average square of the energy variations is proportional to the integral J_0,

$$J_0 = \frac{2\pi\nu}{\omega(\omega^2 + \nu^2)} + \frac{2\nu^2}{(\omega^2 + \nu^2)^2}\left\{\exp\left(-\frac{2\pi\nu}{\omega}\right) - 1\right\}. \qquad (6)$$

We introduce the dimensionless parameter $\beta = 2\pi\nu/\omega$. For $\beta \ll 1$ (long noise correlations) $J_0 \approx \beta\omega^{-2}$, whereas for $\beta \gg 1$ (short noise correlations) $J_0 \approx 4\pi^2\beta^{-1}\omega^{-2}$. The direct calculation of J_1 yields the expression

$$J_1 = \frac{\nu^2}{(\omega^2 + \nu^2)^2}\left\{\exp\left(-\frac{2\pi\nu}{\omega}\right) - 1\right\}^2, \qquad (7)$$

with asymptotics $J_1 \approx \left(4\pi^2\right)^{-1}\beta^4\omega^{-2}$ for $\beta \ll 1$ and $J_1 \approx \left(4\pi^2\right)^{-1}\beta^{-2}\omega^{-2}$ for $\beta \gg 1$. The higher integrals differ from J_1 only by the presence of the exponential factor:

$$J_{k\geq 1} = J_1 e^{-\beta(k-1)}. \qquad (8)$$

Thus the random process of the energy variation in both limiting cases $\beta \ll 1$ and $\beta \gg 1$ can be seen as the superposition of a noncorrelated process with large intensity and a process of small intensity with exponential correlations, similar to that of the noise. We limit ourselves to treatment of the limiting cases; then the energy kinetics can be basically regarded as an uncorrelated process. The rate of the noise correlation decay ν will disclose itself only in the typical magnitude of the energy variations per period.

The near absence of correlations of energy increments ΔE in consequent oscillation periods looks natural for short noise correlations, but seems puzzling for the long ones. To clarify the issue, let's consider a toy model, in which the force ξ remains constant for intervals of time much longer than the oscillation period $T = 2\pi/\omega$, and occasionally changes instantaneously to a new constant value $\xi' = \xi + \Delta F$. Since the constant force does not change the energy of the oscillator at all, the contributions to the correlator of ΔE will come only from the periods that contain jumps of ξ. If n is a period with a jump, then $\langle \Delta E_n^2 \rangle \propto \langle \Delta F^2 \rangle$, but $\langle \Delta E_n \Delta E_{n+1} \rangle \approx 0$.

Apart from the energy change by the noise, our model (1) includes the process of the energy dissipation governed by the friction. For the sufficiently weak damping during one period of the unperturbed motion the oscillator changes its energy by

$$\delta E = -mA^2\gamma\omega^2 \int_0^{\frac{2\pi}{\omega}} \cos^2\omega t dt = -\left(\gamma\frac{2\pi}{\omega}\right)E. \qquad (9)$$

We can describe the energy kinetics of the oscillator as a random walk on the energy axis with discrete time (measured in the motion periods T) and variable, random magnitude of the step on the energy axis, that is governed by the following mapping:

$$E_{n+1} = E_n\left(1 - \gamma\frac{2\pi}{\omega}\right) + F\sqrt{\frac{2E_n}{m\omega^2}}\varphi(\beta)\,\xi_n, \qquad (10)$$

where $\varphi(\beta) = \sqrt{\beta}$ for $\beta \ll 1$ and $\varphi(\beta) = 2\pi/\sqrt{\beta}$ for $\beta \gg 1$, and ξ_n is a noncorrelated random quantity with $\langle \xi_n \rangle = 0$, normalized by the condition $\langle \xi_n^2 \rangle = 1$.

To study the average time of the excitation of the oscillator from the initial state with the zero energy to the given threshold energy E^* it is convenient to introduce the variable $z = \sqrt{E/E^*}$. Equation for the random walk on the axis of the reduced square root of the energy has the form

$$z_{n+1} = z_n(1 - \varepsilon) + \mu\xi_n, \qquad (11)$$

where ε and μ are the dimensionless parameters

$$\varepsilon = \gamma\frac{\pi}{\omega}, \qquad \mu = \frac{F\varphi(\beta)}{\sqrt{2m\omega^2 E^*}}. \qquad (12)$$

It must be noted that Eqs. (10) and (11) lose their validity for very small values of E and z correspondingly. Formally this loss is displayed by the possibility of negative values of the right hand sides of the equations. Physically this limitation origins in the failure of the perturbation theory that has been used for the calculation of the energy variation in (4). However, the domain of inapplicability is very small ($E < \mu^2 E^*$ or $z < \mu$), and modification of the mappings in this range practically does not influence the global behavior of the system. In the numerical experiments we have used the Eq. (11) in the form $z_{n+1} = |z_n(1 - \varepsilon) + \mu\xi_n|$. This modification has a physical grounding, since the oscillator with zero energy acquires for the first period T the energy of the order $\mu^2 E^*$, that agrees with the modified mapping.

In the absence of damping ($\varepsilon = 0$) the random walk is symmetric: the probabilities to move to higher (right)

or lower (left) sides of the energy axis are equal. For the symmetric random walk between the adjacent sites on a one-sided chain (with the reflecting wall at the zeroth site), that starts on the zeroth site, the average time Θ_0 of the first passage across the site $N \gg 1$ is $\Theta_0 = N^2$ [5]. This result can be applied to our model by replacing the variable shift by a constant one, that equals to the square averaged value $\Delta z = \sqrt{\langle \Delta z^2 \rangle} = \mu$. Thus we obtain

$$\Theta_0 = \mu^{-2}. \tag{13}$$

The numerical calculation shows that this relation holds with accuracy about 1% in the wide range of small values of μ.

In the presence of damping ($\varepsilon > 0$) the random walk loses symmetry: from (11) it follows that the probability of increase of the energy (or z) is less than the probability of its decrease. Obviously that increases the time of the first passage through the threshold energy E^*. We define the lengthening factor $L(\varepsilon, \mu)$ as the ratio of the average times of the first passage in the system with damping and without it:

$$L(\varepsilon, \mu) = \frac{\Theta(\varepsilon, \mu)}{\Theta_0}. \tag{14}$$

For the random walk between adjacent sites on the one-sided chain, in which the probabilities of jumps form the site j to the right (p_j) and to the left (q_j) are not equal, the average time of the first passage through the site N by the walker that starts from the zeroth site, was calculated by Murthy and Kehr [6]. It is given by the expression

$$\Theta = \sum_{k=0}^{N-1} \frac{1}{p_k} + \sum_{k=0}^{N-2} \frac{1}{p_k} \sum_{i=k+1}^{N-1} \prod_{j=k+1}^{i} \frac{q_j}{p_j}. \tag{15}$$

Let's consider the case in which the probabilities depend on the numbers of sites linearly:

$$p_k = \frac{1}{2} - \frac{\Delta}{N} k, \quad q_k = \frac{1}{2} + \frac{\Delta}{N} k. \tag{16}$$

Then for the lengthening factor we obtain from (15)

$$L(\Delta N) \approx \frac{e^{2\Delta N} - 1 - 2\Delta N}{2(\Delta N)^2} \tag{17}$$

We apply this result to the model (11). At every step the change in z consists of the symmetric random shift with magnitude $\mu\xi$ and of the systematic negative shift due to relaxation that equals $-\varepsilon z$. The probability P_+ that the eventual shift will be positive is given by the expression

$$P_+ = \int_{\varepsilon z \mu^{-1}}^{\infty} W_\xi(x)dx = \int_0^{\infty} W_\xi(x)dx - \int_0^{\varepsilon z \mu^{-1}} W_\xi(x)dx, \tag{18}$$

where $W_\xi(x)$ is the distribution function of the random quantity ξ. This distribution is an even function, therefore the first integral equals 1/2. Since $z \le 1$ and $\varepsilon \ll \mu$, the function in the second integral can be replaced by $W_\xi(0)$. Thus we obtain

$$P_+ \approx \frac{1}{2} - \frac{\varepsilon}{\mu} z W_\xi(0). \tag{19}$$

From the comparison of this equation with the expression for p_k in (16) it can be seen that the asymmetric random walk (11) with variable steps can be approximated by the asymmetric random walk with steps of the constant length μ with random signs, that is by the model (16) with the parameter values

$$\Delta = \frac{\varepsilon}{\mu} W_\xi(0), \qquad N = \frac{1}{\mu}. \tag{20}$$

The last expression agrees also with Eq. (13).

The lengthening factor depends on the parameter ΔN, that is proportional to $\kappa = \varepsilon \mu^{-2}$.

The following graph shows the dependence of the logarithm of the lengthening factor, $\ln L$, on the parameter κ for the normal (gaussian) distribution of ξ ($W_\xi(0) = 1/\sqrt{2\pi} = 0.399$).

It can be seen that it is a nonlinear function that starts with a linear asymptotic with a small inclination and then approaches a linear dependence with the slope about three times steeper (cf. (17)).

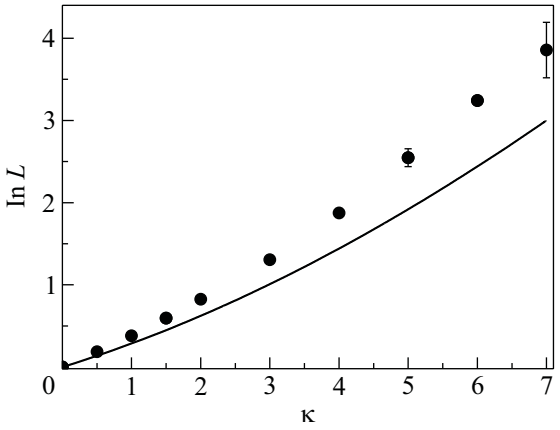
Our estimate of the rate of passage through the energy threshold ($\Gamma = \Theta^{-1}$) has the form

$$\Gamma = \frac{\omega}{2\pi} \mu^2 \exp(-\#\kappa). \tag{21}$$

where # is a numerical constant about unity (from the numerical calculations # = 0.83, see Figure). For the model (1),(2) with the exponentially correlated noise with short correlations ($\beta \gg 1$) we obtain

$$kT = \frac{F^2}{m\gamma\nu}. \tag{22}$$

Eventually the exponential factor obtains the form of the Kramers exponent (3) but with a numerical coefficient #: $\Gamma \propto \exp(-\#E^*/kT)$. We note that the numerical value of # is not universal: it depends on the $W_\xi(0)$,



Dependence of the logarithm of the lengthening factor L on the parameter $\kappa = \varepsilon\mu^{-2}$, calculated from Eq. (14) (solid line) and obtained in the numerical experiment (solid circles)

and consequently depends on the form of the distribution of ξ. Thus the energy kinetics under the influence of coloured noise depends not only on the intensity of noise. On the other hand, the similarity between the exponents in (3) and (21) is not trivial: by accepting Eq. (4), that is valid only for $F \ll m\omega^2 A$, we have excluded the transition to the white noise limit, that is defined by the limit $\nu \to \infty$, $F \to \infty$, $F^2/\nu = \text{const}$.

In the opposite limit of long correlations ($\beta \ll 1$) the quantity in the exponent of (21) increases proportionally to the correlation time of the noise $\tau_c = \nu^{-1}$ (cf. (2)). This dependence for the rate of transitions between equilibrium states of a bistable system under the influence of the colored noise was derived theoretically (by different method) and confirmed experimentally in [7].

The exponential factor in (21) resembles the result obtained from the kinetic equation of the Fokker-Planck type modified for coloured noise. For the harmonic oscillator in the absence of the external driving force we can rewrite kinetic equation obtained in [8–10] using the following relations.

The oscillator energy E and coordinate x can be expressed through the new variables a and a^*:

$$E = \omega a a^*, \quad x = \frac{1}{\sqrt{m\omega}}(a + a^*). \tag{23}$$

The equation of motion for a has the following form:

$$\dot{a} = -i\omega a - \varepsilon\omega a + \tilde{\xi}F, \tag{24}$$

where

$$\frac{\xi}{\sqrt{m\omega}} = \tilde{\xi} + \tilde{\xi}^*. \tag{25}$$

The probability distribution function P satisfies the kinetic equation

$$\frac{\partial}{\partial t}P = \frac{\partial}{\partial E}(\varepsilon E P) + \frac{\partial}{\partial E}\frac{F^2}{m\omega}\tilde{D}(E)\frac{\partial}{\partial E}P, \tag{26}$$

where

$$\tilde{D}(E) = (\beta + \varepsilon)\frac{E}{\omega}. \tag{27}$$

From this equation one can estimate the transition rate

$$\Gamma \sim \exp\left[-\frac{\kappa\varphi^2(\beta)}{\beta + \varepsilon}\right]. \tag{28}$$

In this expression # in (21) is replaced by $\varphi^2(\beta)/(\beta+\varepsilon)$ which is of the order of unity for $\varepsilon \ll \beta \ll 1$.

The pre-exponential factors in Kramers' theory depend on parameters of the potential of the unperturbed system and on the damping coefficient γ. Our calculation produces the pre-exponent that depends on the parameters of noise, does not include γ at all and correctly describes the case of zero friction.

The authors are grateful to L.V. Keldysh for useful discussions. The authors acknowledge the support by the "Russian Scientific Schools" program (grant # NSh-4464.2006.2), the Russian Foundation for basic research and Russian Ministry of Science grant # 8-02-01020-A.

1. M. I. Dykman, Sov. Phys. JETP **64**, 927 (1986).
2. M. Lax, Rev. Mod. Phys. **38**, 359 (1966).
3. H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).
4. G. Margolin and E. Barkai, Phys. Rev. E **72**, 025101(R) (2005).
5. I. Goldhirsch and Y. Gefen, Phys. Rev. A **33**, 2583 (1986).
6. K. P. N. Murthy and K.W.Kehr, Phys. Rev. A **40**, 2082 (1989).
7. P. Hanggi, T. J. Mroczkowski, F. Moss, and P. V. E. McClintock, Phys. Rev. A **32**, 695 (1985).
8. N. S. Maslova, R. Johne, and N. A. Gippius, Pis'ma v ZhETF **86**, 135 (2007).
9. R. Johne, N. S. Maslova, and N. A. Gippius, Solid State Commun. **149**, 496 (2009).
10. N. S. Maslova, R. Johne, and N. A. Gippius, Pis'ma v ZhETF **89**, 718 (2009).