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Black hole motion in entropic reformulation of general relativity

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We consider a system of black holes – a simplest substitute of a system of point particles in the mechanics of general relativity – and try to describe their motion with the help of entropic action: a sum of the areas of black hole horizons. We demonstrate that such description is indeed consistent with the Newton's laws of motion and gravity, modulo numerical coefficients, which coincide but seem different from unity. Since a large part of the modern discussion of entropic reformulation of general relativity is actually based on dimensional considerations, for making a next step it is crucially important to modify the argument, so that these dimensionless parameters acquire correct values.

A recent paper [1] by Eric Verlinde attracted a new attention to the old idea of entropic reformulation of general relativity [2]. Naturally, this paper provoked an avalanche of new publications, we list just a few in [3]. In this letter we provide still another small illustration of how this idea could work, and what kind of discrepancies could arise in attempts to formulate and validate it at quantitative level.

1. The problem and simplifying assumptions. In ordinary approach to quantum gravity one integrates over gravitational fields $\{g\}$ with the weight defined by Einstein-Hilbert (or Palatini) action $S\{g\}$ and some measure $d\mu\{g\}$:

$$Z = \int e^{S\{g\}/\hbar} d\mu\{g\}. \tag{1}$$

Different formulations of quantum gravity, from superstring theory to loop gravity, make use of different realizations (and, perhaps, generalizations) of $S\{g\}$. At the same time, quantum gravity is sometime believed to be a *topological* theory, and topological theories are those which do not have a non-trivial action, $S = 0$, only the measure $d\mu$. This motivates the search for a *reformulation* of general relativity, where Einstein action will be substituted by some measure $d\mu$, which hopefully will be of pure geometric nature. From the point of view of thermodynamics, $Z = e^{F/T}$, and $dF = TdS - dE$, where the first term, TdS, is associated with the measure $d\mu$, while the second, dE,– with the action S. In topological theories $dE = 0$ and $Z = e^S$ is defined by pure *entropic* considerations. Thus, if gravity is believed to be a topological theory, it is actually believed to be pure entropic: instead of Einstein action one can use just the entropy function.

Usually the simplest system to formulate and explore the dynamical principles of the theory is a collection of point particles. In the case of gravity there are no point particles: the simplest objects which exist in this theory are black holes. Thus the simplest toy model in gravity is a collection of black holes. The entropy function for this system is defined by the sum of areas of the black hole horizons, i.e. *entropic action* is simply

$$S = \varkappa\hbar \cdot L_{Pl}^{2-D} \sum_i A_i, \tag{2}$$

where A_i is the area of the horizon of the i-th black hole, $M_{Pl} = L_{Pl}^{-1}$ is the Planck constant and the measure $d\mu$ is now trivial. We write the action in D space-times dimensions, to have one extra parameter, which can enter expressions for dimensionless coefficients in what follows.

Of course, in entropic formulation there is no space-time, thus D is no more than a free parameter of the theory. Moreover, there is also no Plank constant $\hbar$ in $e^{S/\hbar}$, all dynamics in topological theory is pure *combinatorial*: dictated by counting of degrees of freedom (basically a calculus of integers, a section in *number theory*, perhaps, subjected to one or another regularization when D exceeds 2). The minimal action principle is still applicable in the classical approximation, when S is large, i.e. when all the distances, i.e. Schwarzschild radia and distances between the black holes are large as compared to the Planck length L_{Pl}. In what follows we keep $L_{Pl} = 1$ to simplify the formulas.

To promote the geometric formula (2) into a real dynamical principle one needs to specify, how the areas A_i depend on the state of the system, i.e. on location of our black holes and their velocities. Different versions of gravity theories (say, different modifications of Einstein-Hilbert or Palatini actions) can provide different expressions, but all of them are quite sophisticated and non-local. They are, however, drastically simplified for remote black holes, when distances between them are much bigger then their Schwarzschild radia, and this is the case that we are going to analyze in the present letter.

Moreover, we make two further simplifying assumptions. First, we define the horizons by Laplace's principle: that the second cosmic (parabolic) velocity is equal to the light speed $c = 1$. Second, most disputable, we assume that the shape of the horizon of a moving black hole is deformed by Lorentz contraction in the longitudinal direction (e.g. a spherical horizon for an isolated black hole becomes an axially symmetric ellipsoid). These assumptions make calculations trivial and provide a clear picture of what happens without going deep into sophisticated analysis of non-linear gravity.

2. Kinematics: the second Newton's law. In general relativity our particles (black holes) are never at rest: there is no parameter (like large mass) which could be adjusted to keep them in a given position. Moreover, they are necessarily accelerated. Thus the first question to ask is where acceleration is in the action principle (2). The answer is already suggested in [1]: for every particular *probe* black hole we have a kind of a Newton's second law,

$$m\mathbf{a} = -\varkappa T\,\frac{\delta A}{\delta \mathbf{x}}, \tag{3}$$

where m is the mass, T the temperature (inverse of Schwarzschild radius r), A the area $A \sim r^{D-3}$, $\varkappa A$ the entropy, and x, v and a are position, velocity and acceleration of the black hole.

We, however, prefer to interpret/derive the "Newton law" (3) in a somewhat different way from [1]: just as a simple kinematical relation. Namely, imagine that our particle (black hole) just started to move. Then during the time δt it passes the distance

$$\delta x = \frac{a\delta t^2}{2}, \tag{4}$$

then

$$a\delta x = \frac{v^2}{2} \tag{5}$$

is expressed through velocity $v = a\delta t$ that it finally achieved. At the same time, the horizon of the black hole is now deformed by the Lorentz contraction: instead of a sphere with the area $A = \pi r^2$ it is now an ellipsoid with the smaller area

$$A + \delta A = A(1 - C_{LC}v^2). \tag{6}$$

Thus

$$a\delta x = \frac{v^2}{2} = -\frac{1}{2C_{LC}}\cdot\frac{\delta A}{A} = -C_{NL}\frac{\varkappa T\delta A}{m}, \tag{7}$$

where

$$\boxed{C_{NL} = \frac{m}{\varkappa AT}\cdot\frac{1}{2C_{LC}}}. \tag{8}$$

If black hole was already moving then v is not infinitesimally small, and $\delta\mathbf{x} = \mathbf{v}\delta t$. Still $\mathbf{a}\delta\mathbf{x} = \mathbf{a}\mathbf{v}\delta t = \mathbf{v}\delta\mathbf{v}$, while $\delta A = -2C_{LC}A\mathbf{v}\delta\mathbf{v}$, so that (7) is preserved with the same value of C_{NL}. This is consistent with (3) – though is a somewhat weaker, a scalar rather then a vector relation – up to numerical constant C_{NL}, which still needs to be evaluated.

Evaluation of C_{NL} seems quite important. The point is that (3) can actually be written on pure dimension grounds – provided one wants to find *some* relation of this kind. What could take us further, beyond pure dimensional consideration, and thus provide a real *quantitative* argument in support of entropic-reformulation ideas, is evaluation of dimensionless numerical coefficients. This is what makes any practical way to calculate C_{NL} so interesting. Of course, in the case of our simple model this is straightforward: to find C_{NL} we need to know the Lorentz-contraction factor C_{LC} and the ratio $\varkappa AT/m$. This will be the subject of the sections 4 and 5 below.

Note that there was no reference to the action principle (2) in this section: Eq.(7) is a pure kinematical relation.

3. Dynamics: the Newton's gravity law. The action plays role when we study the interaction of several black holes. Then there are two competing effects. First, black hole horizon is deformed in the presence of gravitational field of the other black holes, this leads to increase of the area. Second, the field accelerates the black hole, what decreases the area due to Lorentz contraction. The minimal action principle requires that the two effects exactly compensate each other.

Laplace principle easily defines the horizon of two black holes at rest:

$$\frac{m_1}{|\mathbf{r}-\mathbf{x}_1|^{D-3}} + \frac{m_2}{|\mathbf{r}-\mathbf{x}_2|^{D-3}} = C_{\rm pot}^{-1} \tag{9}$$

C_{pot} is a D-dependent Newton's constant, whose exact value is irrelevant in consideration of this section.

If the distance R between the black holes is much bigger than their Schwarzschild radia then in the first approximation we get

$$\frac{m_1}{r_1^{D-3}} = C_{\rm pot}^{-1} - \frac{m_2}{R^{D-3}} \tag{10}$$

i.e.

$$A_1 {=} \Omega_{D-2}r_1^{D-2} = \Omega_{D-2}\left(\frac{m_1}{C_{\rm pot}^{-1} - m_2/R^{D-3}}\right)^{\frac{D-2}{D-3}}. \tag{11}$$

The closer the black holes the bigger are their horizons.

Under a small shift $\delta \mathbf{x}_1$ of the black hole in space its horizon area changes:

$$\delta A_1\left((D-2)\frac{\delta r_1}{r_1} - 2C_{LC}(\mathbf{a}_1\delta\mathbf{x}_1)\right) = 0. \tag{12}$$

The second item is the effect of Lorentz contraction and the sum of two terms vanishes on equation of motion for (2). We assume here that the equations of motion for different black holes are fully separated.

From (10) we have:

$$\frac{m_1}{r_1^{D-3}}\frac{\delta r_1}{r_1} = -\frac{m_2}{R^{D-2}}\delta R, \tag{13}$$

where $R\delta R = \mathbf{R}\delta\mathbf{x}_1$. Thus (12) is consistent (again up to a difference between scalar and vector equations) with the Newton's gravity law:

$$\mathbf{a}_1 = -C_{GL}\boldsymbol{\nabla}_1\left(\frac{C_{pot}m_2}{R^{D-3}}\right) \tag{14}$$

with

$$\boxed{C_{GL} = \frac{D-2}{D-3}\cdot\frac{1}{2C_{LC}}}. \tag{15}$$

4. Black hole numerology. While C_{GL} in (15) depends on nothing but the Lorentz contraction factor c_{CL}, the kinematical factor C_{NL} in (8) is different: it involves detailed information about the black hole physics [4–8].

As explained at the end of section 1, we define the Schwarzschild radius r by equating the parabolic velocity to $c=1$, i.e. from the condition

$$C_{pot}\frac{m}{r^{D-3}} = 1. \tag{16}$$

We already used a more involved version of this equation in (9).

Parameter $C_{\rm pot}$ is normalization of a Green function for Laplace equation in $D-1$ dimensions, $\Delta_{D-1}\left(C_{\rm pot}\frac{m}{r^{D-3}}\right) = C_L m\delta^{(D-1)}(\mathbf{r})$. The Gauss law then implies, that

$$C_{pot}(D-3)\Omega_{D-2} = C_L. \tag{17}$$

The Hawking temperature is

$$T = C_T/r \tag{18}$$

where C_T can be defined from quasiclassical considerations [5], a simple version of such derivation is recently analyzed in [8].

The area of the spherical horizon in the rest frame of the black hole is

$$A = \Omega_{D-2}r^{D-2}, \tag{19}$$

where the angular integral

$$\Omega_{D-2} = \frac{2\pi^{\frac{D-1}{2}}}{\Gamma\left(\frac{D-1}{2}\right)}. \tag{20}$$

Finally, the entropy of the black hole is proportional, which presumably enters the r.h.s. of (3), is proportional but not equal to the area,

$$\text{Entropy} = S/\hbar = \varkappa A. \tag{21}$$

In terms of these parameters the ratio

$$\kappa ATm = \varkappa C_T C_{pot}\Omega_{D-2} = \frac{\varkappa C_T C_L}{D-3}. \tag{22}$$

The values of the parameters for $D=4$ are known since [5]:

$$\begin{gathered} D=4: \quad C_{\rm pot}=2, \quad \varkappa = \frac{1}{4}, \quad C_T = \frac{1}{4\pi}, \\ \Omega_2 = 4\pi \implies \frac{\varkappa AT}{m} = \frac{1}{2}. \end{gathered} \tag{23}$$

Generalization to arbitrary D is now available in many papers, see, for example, [6]. We borrow concrete formulas from a recent review [7]:

$$\begin{gathered} C_{\rm pot}\Omega_{D-2} = \frac{16\pi}{D-2}, \quad \varkappa = \frac{1}{4}, \\ C_T = \frac{D-3}{4\pi} \implies \frac{\varkappa AT}{m} = 4\varkappa\cdot\frac{D-3}{D-2} = \frac{D-3}{D-2}. \end{gathered} \tag{24}$$

Substituting this ratio into (8), we obtain:

$$\boxed{C_{NL} = \frac{D-2}{D-3}\cdot\frac{1}{2C_{LC}} \overset{(15)}{=} C_{GL}}. \tag{25}$$

Remarkably, the two coefficients C_{NL} and C_{GL}, which both need to be unities for entropic principle to work, at least coincide for arbitrary space-time dimension D.

5. The Lorentz-contraction factor and parameters C_{NL} and C_{GL}. To check if the common value of the two parameters is unity or not, we need to evaluate the Lorentz-contraction factor C_{LC}. It defines the deviation relative area of the surface of the axially symmetric ellipsoid $x_1^2+\ldots+x_{D-2}^2+\gamma^2z^2=1$ with $\gamma^{-1}=\sqrt{1-v^2}$ and Ω_{D-2} from unity for $v^2\ll 1$:

$$2C_{LC} = \frac{\int_0^\pi \sin^{D-1}\theta d\theta}{\int_0^\pi \sin^{D-3}\theta d\theta} = \frac{D-2}{D-1}. \tag{26}$$

Substituting this value into (25) we finally obtain:

$$\boxed{C_{NL} = C_{GL} = \frac{D-2}{D-3} \cdot \frac{1}{2C_{LC}} = \frac{D-1}{D-3} \neq 1}. \quad (27)$$

6. Conclusion. The main goal of this letter was explicit evaluation of two numerical coefficients: C_{NL} and C_{GL} in the second (7) and gravitation (15) laws respectively. For entropic reformulation of general relativity to work in its most naive form, based on the action principle (2), these two coefficients should be equal to unity. We did not manage to adjust them in this way, moreover, our answers depend non-trivially on the free parameter D – the space-time dimension. Remarkably, though not unities, the two coefficients are the same, and both discrepancies can be simultaneously cured if one slightly changes the definition of the Lorentz-contraction factor, from (26) to $(D-2)/(D-3)$, for example, by *postulating* the velocity-dependence of the action (2) in the form $A_{\text{rest}}\left(1 - \frac{D-2}{D-3} \cdot \frac{v^2}{2} + O(v^4)\right)$. Such *ad hoc* postulates would, however, decrease the attractiveness of the entire approach and therefore are undesirable. Before one can move further with *quantitative* development of entropic reformulation along the lines of our section 1, which would include gravitational radiation and corrections beyond classical (small L_{Pl}) approximation, and their comparison with various programs of gravity quantization, it is necessary to find and correct the mistakes (arithmetical or conceptual) in the simple calculations, described in above sections 2-5. The next small step would be to consider the next corrections in v/c and r/R, also including non-linear effects of general relativity (like those, responsible for perihelion shift and the Lamb shift of orbital frequencies [9]).

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