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Towards construction of geometric bosonic quantum field theories I

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We present a geometric construction of 2D chiral boson theories on manifolds with tangent bundle admitting flat connection with possible torsion. The construction is based on embedding of the theory into the supersymmetric $\beta\gamma - bc$ system. To perform the embedding we do a covariant smooth point-splitting of vector field observables and then the fermions are killed by a strongly oscillating chiral gauge transformation.

1. Motivation. In the traditional perturbative QFT one defines a quantum theory using Gaussian functional integrals and treating interactions perturbatively. This approach does not give a non-perturbative definition of the theory. For instance, in the non-Abelian gauge theories one approximates the non-Abelian fields with Abelian ones when making a perturbative expansion. It is thus desirable to account for the non-linearity exactly when formulating the quantum theory.

Two-dimensional sigma models may be considered as toy models for gauge theories, and non-linearity is expressed in the sigma-models through the absence of global linear structure on the target manifold. Some 2D sigma-models for target spaces without linear structure (e.g. a group manifold) can be successfully formulated in the axiomatic current-algebra approach [1, 2], but it is not clear how to generalize such approach to include instantons.

Recently a class of supersymmetric theories was considered that allows for non-linear formulation. These are the non-linear geometric (or "instantonic") quantum field theories in 1,2 and 4 dimensions constructed and studied in the works [3 – 5], see also [6]. These theories correspond to supersymmetric Morse quantum mechanics, supersymmetric $\beta\gamma - bc$ field theory and $\mathcal{N} = 2$ twisted super-Yang-Mills theory at instantonic point. All these theories may be formulated geometrically by exact localization to finite-dimensional space – the moduli space of generalized instantons. This allows to account exactly for the non-linearity.

In this letter we study the question of extending the above non-linear formulation to non-supersymmetric theories. To do this we want to see the bosonic theory inside the geometric supersymmetric theory.

This idea is quite natural since supersymmetric theories are known to have better UV properties,[1)] hence considering non-supersymmetric theories as softly-broken supersymmetric ones gives an improved non-perturbative version of Pauli-Villars regulators. Equivalently, this means that one maps the observables of the bosonic theory to observables of a supersymmetric theory, since states may be created by observables.

A natural way to perform such a mapping is to take a bosonic correlation function and put an extra observable, corresponding to a large mass term for fermions (i.e. soft-breaking term). In this letter we show how this *general* recipe works and for the case of target space admitting flat tangent bundle (with possible torsion) we arrive at the chiral boson theory equivalent to the one formulated in axiomatic approach.

We study the problem in 1 and 2 dimensions. In particular, in 1 dimension we want to define bosonic correlators in *geometric* supersymmetric quantum mechanics with action

$$S = -i \int dt \left(p_i \frac{\partial}{\partial t} X^i - \pi_i \frac{\partial}{\partial t} \psi^i \right) \quad (1)$$

and in 2 dimensions we want to see the non-linear theory of chiral [2)] boson inside the *geometric* $\mathcal{N} = (2,2)$ supersymmetric $\beta\gamma - bc$ field theory with action

$$S = -i \int_\Sigma d^2 z (p_i \bar\partial X^i - \pi_i \bar\partial \psi^i + \bar{p}_{\bar i} \partial \bar{X}^{\bar i} - \bar\pi_{\bar i} \partial \bar\psi^{\bar i}) \quad (2)$$

We follow here the notations of [3 – 5] ($\beta_i = p_i$, $\gamma^i = X^i$, $b_i = \pi_i$, $c^i = \psi^i$).

This study is partially motivated by applications to elucidating the pure spinor approach to superstring theory: non-supersymmetric "curved $\beta\gamma$-systems" play an important role in this approach [7].

In case of target manifolds with linear structure, the fermions ψ^i and π_i correspond to dX^i and $\iota_{\partial/\partial X^i}$ and

[1)] Geometric supersymmetric theories are even stronger: they are finite and allow for a non-linear formulation.

[2)] Actually, the resulting theory may occur non-chiral for target manifolds not admitting flat tangent bundles.

one may get the bosonic theory simply by crossing out the fermions π and ψ in the above supersymmetric theory, but otherwise the fermions do not have any global coordinate-independent meaning (in simple terms, the target space coordinate transformations mix p with $\pi\psi$), so it is non-trivial to single out the bosonic observables.

To see a bosonic theory we consider the method of adding a large fermionic mass term of the form $\exp\{\int d^2z\, m\pi(z)\psi(z)\}$. The main obstacle we meet in 2D case is that any mass term of this form is actually a gauge field and can be eliminated by gauge-rotating π and ψ fields, as was pointed out by Nekrasov [8]. We turn this obstacle into a positive feature: we define a mass term as a strongly-oscillating gauge transformation of fermions, so that any observables having non-vanishing fermionic charge start to give strongly-oscillating (in worldsheet position) contribution to correlation function. Smoothing the evaluation-point leads to suppression of fermions. Thus, we introduce a hierarchy of distance scales: small, medium and large: $1/m \ll \epsilon \ll bos$, where $1/m$ is a scale of oscillations due to "mass" for fermions, ϵ is a smooth point-splitting scale and bos is a scale at which we get the bosonic theory. Similarly, the supersymmetric observables corresponding to local vector fields should be made smoothly non-local by covariant point-splitting (here one needs to choose the connection in TX). This procedure singles out the right amount of vector fields (satisfying $Dv_i = 0$), corresponding to "primary" bosonic currents, while the remaining vector fields correspond to composite bosonic observables regularized with chosen point-splitting.

To get the axiomatic bosonic theory for parallelizable manifolds our prescription finally boils down to: make a covariant point-splitting and then cross out all observables having non-zero fermionic charge.

In order to avoid complications, we assume that the volume form used in construction of the bosonic theory is invariant under the action of vector fields that parallelize X.

We hope that such constructions may be generalized to theories with instantons.

2. Overview of geometric theories. Here we review briefly how to deal with geometric (or instantonic) field theories on the example of $\beta\gamma - bc$ theory (2). Such theories are rigorously defined on any almost-complex target manifold by postulating the prescription of localization on the zeroes of the vector field $\bar\partial X$ [3–5]. Fermions $\psi^i(z)$ are identified with $dX^i(z)$ and the simplest "evaluation" observables correspond to differential forms on X. The supercharge acts as a de Rham differential on X: $Q = d_X$. The correlation functions of evaluation observables are computed by pulling them back to the moduli space of solutions to $\bar\partial X = 0$ (or, in general, to zeroes of any vector field on the space of map that we provide) and integrating over it. Closed forms correspond to BPS (topological) sector and their correlators generate the celebrated Gromov-Witten invarints.

The general correlation functions, containing momentum fields (p_i and π_i), correspond to deforming the vector field on the space of maps $\varphi : \Sigma \to X;\ z \mapsto X(z)$ from $\bar\partial X$ to $v_\epsilon = \bar\partial X + \epsilon^\alpha V_\alpha$. Here V_α and $\bar\partial X$ are elements of $\Omega^{(0,1)}(\Sigma)\otimes\varphi^*(T^{(1,0)}X)$. In general, V_α may be any (non-local) vector field depending on maps $\Sigma \to X$ and we need to search for the map φ which for any z solves

$$\frac{\partial}{\partial\bar z}X(z)d\bar z + \epsilon^\alpha V_\alpha(z,\varphi) = 0. \tag{3}$$

The evaluation observables are then localized to the solution of (3). Acting on the resulting correlator with Lie derivative $\mathcal{L}_{\partial/\partial\epsilon^\alpha}$ and then setting $\epsilon^\alpha = 0$ we get $\mathcal{O}_{V_\alpha}$ observable, or, acting with substitution $\iota_{\partial/\partial\epsilon^\alpha}$, we get π_{V_α} observable – these observables thus correspond to infinitesimal deformations. Since when paired with deformation by V_α these observables give a number, they are naturally the $(1,0)$-forms on Σ (natural pairing assumes integration over Σ).

A special case of these general deformation observables is when the vector field $V(\varphi)$ on maps is induced by a vector field $v \in \Gamma(TX)$ on the target manifold X and a $(0,1)$-form on Σ . If we simply take

$$V_\alpha(\varphi,z) = v_\alpha(X(z))\iota_{\partial/\partial z}\delta^{(2)}(z - z_v) \tag{4}$$

it would be a local vector field. In coordinate chart the corresponding observables are

$$\mathcal{O}_{v_\alpha}(z_v) = ip_i(z_v)v^i_\alpha(X(z_v)) - i\pi_i(z_v)\partial_j v^i_\alpha(X(z_v))\psi^j(z_v) \tag{5}$$

$$\pi_{v_\alpha}(z_v) = i\,\pi_i(z_v)v^i_\alpha(z_v). \tag{6}$$

The more general observables that may be constructed from the vector field on X and some parallel transport operator $T(z,z')$ on $\varphi^*\Gamma(TX)$ and some forms $\omega^{(1,1)}_z$ and $\omega^{(0,1)}$ on Σ correspond to vector fields on the space of maps given by

$$V_\alpha(\varphi,z) = \int_{z'} \omega^{(0,1)}(z)\,\omega^{(1,1)}_z(z')\,T(z,z')\,v_\alpha(\varphi(z')) \tag{7}$$

here $T(z,z')$ is some parallel transport from $X(z')$ to $X(z)$. This vector field reduces to the local one (4) if $\omega^{(0,1)}(z) = \iota_{\partial/\partial z}\delta^{(2)}(z - z_v)$ and $\omega^{(1,1)}_z(z') = \delta^{(2)}(z - z')$, otherwise, we get a class of natural non-local geometric regularizations of local observables which we would use below to define the low-energy theory.

3.Constructing a bosonic theory.

3.1 *One-dimensional case: quantum mechanics.* Consider as a toy model the SUSY QM (1) on the interval $t \in [0,1]$ with free boundary conditions and with real target $\mathbb{R}^n$. In Hamiltonian formalism the states are differential forms on $\mathbb{R}^n$ and the Hamiltonian is zero.

Consider a large mass term $im\psi^i\pi_i$ for fermions, where $m > 0$ is a constant 1-form on the worldsheet, we remark that $n_f = i\psi^i\pi_i = dX^i\iota_{\partial/\partial X^i}$ is a coordinate-independent fermion number (form degree) operator. To saturate the fermionic zero-modes we insert a volume form on X at $t = 1$. This mass term may be represented as Hamiltonian $m\,n_f$, and at time intervals $t \gg 1/m$ the evolution operator e^{-tH} acts a projector to 0-forms, so any fermionic correlation functions at distances $\gg 1/m$ are exponentially suppressed.

Let us now represent the basic set of observables of the low-energy "bosonic" theory as observables in SUSY theory. These are the function evaluation observables and vector-field observables. We assume that other bosonic observables may be constructed by fusing the basic ones, but at distances still larger then $1/m$. Fermionic correlators are screened at large distances while the same-point $i\,\psi\,\pi$ gives a fermion number, which is zero or some number if we change the ordering. So, we can cross out fermions (except for the volume form). Now we can unwind the fermionic gauge transformation, it would remove the "mass term", but otherwise would change nothing since bosonic observables are blind to this transformation. It shows that the recipe to cross out fermions gives a consistent embedding into supersymmetric theory.

For example, the local $\mathcal{O}_{v(X)}$ observable in SUSY theory corresponds in Hamiltonian and geometric formalisms to Lie derivative $\mathcal{L}_v$ acting on wave-forms. Taking it to the low-energy bosonic theory results in the same $\mathcal{L}_v$ (since it has only same-point $\pi\psi$ fermions), but now acting on functions, which is obviously coordinate-invarint.

3.2 *Imaginary mass.* Let us consider the imaginary mass m, this is motivated by 2D case, see below. This mass term is equivalent to the oscillating gauge transformation:

$$\psi(t) \to \psi(t)\,e^{mt} \;\;\&\;\; \pi(t) \to \pi(t)\,e^{-mt}. \qquad (8)$$

The new fermionic propagator of the massive theory is easily related to the massless one:

$$\langle \pi(t)\psi(t')\rangle_m = \langle \pi(t)\psi(t')\rangle e^{-m(t-t')}. \qquad (9)$$

The partition function is unchanged. To kill fermions, the bosonic observables should be mapped to the "smoothened" ones in the SUSY theory. Smoothing is very natural for constructing observables of low-energy theory.

3.2.1. *Smoothened observables.* As a first attempt we try taking geometric observables, corresponding to differential forms $\omega_i\psi^i$ or vector fields $\mathcal{O}_v$ at some point t and then smoothen the evaluation point: $\mathcal{O}(t) \longrightarrow \longrightarrow \int w_\epsilon(t'-t)\mathcal{O}(t')dt'$. Since the mass is equivalent to a strongly fluctuating gauge transformation for ψ and π, convolving this gauge transformation with a smooth function leads to vanishing of fermion correlations.

For example, with a Gaussian smoothing

$$w_\epsilon(t'-t) = \frac{1}{\sqrt{\pi}\epsilon} e^{-|t'-t|^2/\epsilon^2}$$

the smoothened fermionic correlator tends rapidly to zero when

$$|m| \gg \epsilon \;\text{ as }\; \langle \pi(t)\psi_\epsilon(t')\rangle \approx e^{-\frac{1}{2}|m|^2\epsilon^2} G(t,t'),$$

where G is of order of unity. The problem arises with local $\mathcal{O}_v$ current, containing π and ψ at the same point! To eliminate fermions, we need to smoothly point-split π and ψ, so that the microscopic vector field observable becomes non-local.

In geometric formalism we map a local vector field on the space of maps to the non-local one. For that we need to introduce additional data: the connection on the target X, and insert a parallel transport operator T, which is a transport from $X(t)$ to $X(t')$ along the image of the worldsheet:

$$T^i_j(X(t),X(t')) = P\exp\int_{X(t)}^{X(t')} \Gamma\partial_t X\,dt, \qquad (10)$$

with left-to-right ordering. We assume for simplicity that TX admits a flat connection (with possible torsion). Then we map a local vector field on $\Sigma \to X$, induced by vector field v on X, to the non-local one as in eq.(7). In one dimension ω, ω_t are 1-forms and we choose them to have support ϵ with hierarchy of scales $1/m \ll \epsilon \ll bos$.

The corresponding smooth $\mathcal{O}_V$ observable is given in some coordinates by[3)]

$$-\,i\,\mathcal{O}_V(t) = \qquad (11)$$

$$= \int dt' p_i(t)T^i_j(X(t),X(t'))\,v^j(X(t'))\,w(t-t') -$$

[3)]We set $\omega(t) = \delta(t)$ and $\omega_t(t') = w(t-t')$.

$$\int dt'\pi_i(t)T^i_j(X(t),X(t'))(\partial_k v^j(X(t'))\psi^k(t')w(t-t')+$$
$$\int dt'\pi_i(t)\Gamma^i_{ki'}(t)\psi^k(t)T^{i'}_j(X(t),X(t'))v^j(X(t'))w(t-t')-$$
$$\int dt'\pi_i(t)T^i_{j'}(X(t),X(t'))\Gamma^{j'}_{kj}(t')\psi^k(t')v^j(X(t'))w(t-t').$$

When introducing the mass, this becomes the regulated bosonic observable.

3.3. *Two-dimensional case: Chiral boson.* Consider the geometric theory corresponding to action (2). Now any $m\pi\psi$ mass term is generated by a gauge transformation:

$$\psi^i \to \psi^i U(z,\bar z) \quad \& \quad \pi_i \to \pi_i U^{-1}(z,\bar z) \tag{12}$$

for example, one can take $U = e^{2\,\mathrm{Im}(m\bar z)}$ or something similar that satisfies some chosen boundary conditions on the worldsheet. So, the mass term can be considered geometrically as a gauge transformation U.

Analogously to quantum mechanics with imaginary mass, a reasonable smoothing of observables kills the fermions. To construct the geometric representation of bosonic holomorphic current observable, we exactly repeat the construction (11) in two dimensions.

The vector fields satisfying the condition

$$Dv = 0 \tag{13}$$

generate the primary bosonic currents: there are *no quadratic poles in the effective low-energy OPE of such currents and they require no subtractions to give finite correlators.* To prove this claim note that the smoothened supersymmetric current looks as in Eq.(11) (with t replaced by z). Any such current behaves as "primary" in the geometric theory. But now we introduce the mass that kills the second and the last terms in Eq.(11) at large bosonic scales. If v satisfies the Eq.(13) then terms 2 and 4 cancel each other, and so the mass actually does nothing[4)]. Hence for such bosonic currents the OPE is the same as in the supersymmetric theory:

$$\mathcal{O}_{v^i}(z)\mathcal{O}_{v^j}(w) \sim \frac{1}{z-w}\mathcal{O}_{[v^i,v^j]}(w) \tag{14}$$

For example, if the connection is induced by changing coordinates from those in which connection was zero: $\Gamma^j_{kl} = -\partial_k g^j_{l'}(g^{-1})^{l'}_l$, where g is a Jacobian for change of variables; then the set of solutions to Eq.(13) is given by vector fields: $v^j_{(a)} = g^j_a$.

3.4 *Theory on the group manifold.* As an example, consider the chiral bosonic theory on the complex group. Consider holomorphic left-invarint vector fields on the group manifold. Then there is a natural flat connection induced by parallel transport with these vector fields. Making a smoothing with this connection we establish correspondence between the the SUSY currents and the primary bosonic currents.

To saturate the fermionic zero modes, corresponding to translations along X, we insert an invariant volume form, smoothing is not needed for it. The resulting theory, restricted to holomorphic subsector, is then equivalent to first-order bosonic theory on the group manifold with action

$$S = -i\int p_a\omega^a_\mu\,\bar\partial X^\mu, \tag{15}$$

where $\omega^a_\mu\,dX^\mu$ are left-invarint 1-forms on the group manifold.

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4)The same-point operators $\pi(z)\psi(z)$ defined by point-splitting can be made insensitive to gauge-transformation U by putting U^{-1} in the definition of the point-splitting:

$$:\pi_i\psi^j:_{SUSY}(z) = \pi_i(z)U(z,\mathbf{z})\psi^j(z+\epsilon)U^{-1}(z+\epsilon,\mathbf{z}+\bar\epsilon) - \frac{i\,\delta^j_i}{\epsilon}$$

this corresponds to removing of the Quillen anomaly.