Pis'ma v ZhETF, vol. 93, iss. 2, pp. 88 – 91 © 2011 January 25

Goldstone mode relaxation in a quantum Hall ferromagnet due to hyperfine interaction with nuclei

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Submitted 8 December 2010

Spin relaxation in quantum Hall ferromagnet regimes is studied. As the initial non-equilibrium state, a coherent deviation of the spin system from the $\vec{B}$ direction is considered and the breakdown of this Goldstone-mode state due to hyperfine coupling to nuclei is analyzed. The relaxation occurring non-exponentially with time is studied in terms of annihilation processes in the "Goldstone condensate" formed by "zero spin excitons". The relaxation rate is calculated analytically even if the initial deviation is not small. This relaxation channel competes with the relaxation mechanisms due to spin-orbit coupling, and at strong magnetic fields it becomes dominating.

The reported work is aimed at description of the relaxation channel which might become dominant at magnetic fields $B > 10\,\mathrm{T}$ in the so-called quantum Hall ferromagnet (QHF). The latter is a 2DEG state under quantum Hall effect conditions where all electrons of the upper, not completely filled Landau level, have in the ground state spins aligned along $\mathbf{B}$. This spin polarization obviously arises at odd integer fillings: $\nu = 1, 3, ...$ [1]. Besides, experiments and semi-phenomenological theories show that at some fractional fillings, namely at $\nu = 1/3, 1/5, ...$, electrons in the ground state occupy only one spin sublevel, and thereby the fractional QHF state is also realized [2 – 6]. The QHF possesses a macroscopically large spin $\mathbf{S}$ oriented in the direction of the field $\mathbf{B}$ due to negative g-factor in GaAs structures. In the following all calculations are carried out in the form applicable to both odd-integer filling $\nu = 2k+1$ and fractional QHF. This generalization on the $\nu < 1$ case is done in compliance with the well known semi-phenomenological description of the fractional QHF [2, 3] and, in particular, was already used in Ref. [7].

Obviously, there are two different types of initial deviation of the large spin $\mathbf{S}$ from its equilibrium position. The first type represents the case where vector $\mathbf{S}$ is changed in length but its direction is not altered. Then the QHF symmetry is the same as in the equilibrium state. Analysis reveals that this type of initial perturbation is microscopically described by excitation of spin waves where each one corresponds to the spin numbers changed by one: $\delta S = \delta S_z = -1$ [1]. The second type of spin perturbation is a coherent deviation of $\mathbf{S}$ as a whole from the direction $\hat{z} \parallel \mathbf{B}$ without any changes in the length of $\mathbf{S}$. This case means appearance of the Goldstone mode (GM) in the QHF, and microscopically it is described by a "Goldstone condensate" of "zero spin excitons". Every zero spin exciton X_0 represents a change $\delta S_z = -1$ with the total spin kept constant: $\delta S = 0$. For the first type deviation, the relaxation was studied experimentally in Refs. [8 – 10] and theoretically in Refs. [11, 12] and [7]. For the second type, the GM breakdown was theoretically analyzed in the works of Refs. [13] and [14]. (See also Ref. [15].) All these theoretical studies dealt with the relaxation channels where spin non-conservation arose from the spin-orbit (SO) coupling of 2D electrons.

So, as in publications of Refs. [13] and [14], now the considered initial deviation is again of the second type, i.e. starting point is the GM state $|i\rangle = \left(\hat{S}_-\right)^N|0\rangle$, where $|0\rangle$ stands for the QHF ground state, and $\hat{S}_- = \sum_j \hat{\sigma}_-^{(j)}$ is the lowering spin operator. (j labels electrons; $\hat{\sigma}_\pm = (\hat{\sigma}_x \pm i\hat{\sigma}_y)/2$, where $\hat{\sigma}_{x,y,z}$ are the Pauli matrices.) However, in this Letter I report on another relaxation channel where the spin non-conservation is caused by the hyperfine contact coupling to the GaAs nuclei [16 – 19]:

$$\hat{H}_{\rm hf} = \frac{8\pi}{3}\mu_B \sum_n \frac{\mu_n}{I^{(n)}} \left(\hat{\mathbf{I}}^{(n)}\!\cdot\hat{\boldsymbol{\sigma}}\right) \delta(\mathbf{R}-\mathbf{R}_n). \qquad (1)$$

Here μ_B is the Bohr magneton, $\mathbf{R}$ is the electron position, $\hat{\mathbf{I}}^{(n)}$, $\mathbf{R}_n$ and μ_n are spin, position and magnetic dipole moment of the n-th nucleus, respectively. Compared to the SO coupling Hamiltonian, the interaction (1) has one important feature: it violates the translation invariance of the 2D electron system and therefore leads not only to the electron spin non-conservation but also to the non-conservation of the 2DEG momentum. From the viewpoint of the GM breakdown, this means that the hyperfine coupling is sufficient to provide the GM relaxation process without any additional dissipation mechanisms. This property is at variance with the SO interaction relaxation mechanisms. As far as the

SO coupling does not perturb the translation symmetry, the GM relaxation needs additional perturbative interactions providing the momentum and energy dissipation. These in fact are the electron-phonon coupling [13] or interaction with the smooth random potential [14].

The Hamiltonian (1) may be rewritten as [18]

$$\hat{H}_{\rm hf} = \frac{v_0}{2}\sum_n A_n \Psi^*(\mathbf{R}_n)\left(\hat{\mathbf{I}}^{(n)}\hat{\boldsymbol{\sigma}}\right)\Psi(\mathbf{R}_n), \qquad (2)$$

where $\Psi(\mathbf{R})$ is the electron envelope function, and v_0 is the unit cell volume. Both Ga and As nuclei have the same total spin: $I^{\rm Ga} = I^{\rm As} = 3/2$. The parameter A_n, being inversely proportional to v_0, actually depends only on the Ga/As nucleus position within the unit cell. For the final calculation I only need the sum $A^2_{\rm Ga}+A^2_{\rm As}$. If v_0 is the volume of the two atom unit cell, then using values of magnetic moments of the Ga and As stable isotopes [20], I estimate that $A^2_{\rm Ga}+A^2_{\rm As} \approx 4\cdot 10^{-3}\,{\rm meV}^2$.

In order to describe the QHF states, I use again the *excitonic representation* by analogy with previous works [7, 11 – 14]. Namely, by defining the spin exciton creation operator [21]:

$$\mathcal{Q}^\dagger_{\mathbf{q}} = \frac{1}{\sqrt{N_\phi}}\sum_p e^{-iq_x p} b^\dagger_{p+\frac{q_y}{2}}\, a_{p-\frac{q_y}{2}}, \qquad (3)$$

where a_p and b_p are the Fermi annihilation operators corresponding to electron states on the upper Landau level with spin up ($a =\uparrow$) and spin down ($b =\downarrow$), respectively (N_ϕ is the Landau level degeneration number), I consider states $|N\rangle = \left(\mathcal{Q}^\dagger_{\mathbf{0}}\right)^N|0\rangle$ and $|N-1;1;\mathbf{q}\rangle = \mathcal{Q}^\dagger_{\mathbf{q}}\left(\mathcal{Q}^\dagger_{\mathbf{0}}\right)^{N-1}|0\rangle$, where $|0\rangle = |\overbrace{\uparrow,\uparrow,...\uparrow}^{N_\phi}\rangle$ is the ground state at odd-integer filling ν. Both states correspond to the spin z-component $S_z = N_\phi/2 - N$, whereas the total spin number is $S = N_\phi/2$ and $S = N_\phi/2-1$, respectively. [Index p labels intrinsic Landau level states with wave functions $\psi_p(\mathbf{r}) = (2\pi\mathcal{N}_\phi)^{-1/4}e^{ipy}\varphi_k(p+x)$ in the Landau gauge, $\varphi_k(x)$ is the oscillator function; in Eq. (3) and everywhere below we measure lengths in units of l_B and wave vectors in units of $1/l_B$.] The major advantage of these excitonic states is that they are eigen states of the QHF at odd-integer ν:

$$\left[\epsilon_{\rm Z}\hat{S}_z + \hat{H}_{\rm int},\left(\mathcal{Q}^\dagger_{\mathbf{0}}\right)^N\right]|0\rangle = N\epsilon_{\rm Z}|N\rangle,$$
$$\left[\epsilon_{\rm Z}\hat{S}_z + \hat{H}_{\rm int},\mathcal{Q}^\dagger_{\mathbf{q}}\left(\mathcal{Q}^\dagger_{\mathbf{0}}\right)^{N-1}\right]|0\rangle = (N\epsilon_{\rm Z}+\mathcal{E}_q)|N-1;1;\mathbf{q}\rangle, \qquad (4)$$

where $\epsilon_{\rm Z} = g\mu_B B$ is the cyclotron gap, $\hat{H}_{\rm int}$ is the 2DEG Coulomb interaction Hamiltonian, and $\mathcal{E}_q$ is the Coulomb correlation energy of the spin exciton having momentum $\mathbf{q}$ [1]. These equations are accurate to the first order in parameter $r_c = (\alpha e^2/\kappa l_B)/\hbar\omega_c$ considered to be small (ω_c is the cyclotron frequency, $\alpha < 1$ is the averaged form-factor arising due to finiteness of the 2D layer thickness, κ is the dielectric constant). Only small q vectors are relevant to the studied problem (i.e. $ql_B \ll 1$ in common units), and therefore the quadratic approximation for the spin exciton spectrum $\mathcal{E}_q \approx q^2/2M_{\rm x}$ is sufficient. The exciton mass $M_{\rm x}$ is calculated by using the finite thickness form-factor [1, 7, 12], although recently $M_{\rm x}$ was measured experimentally [4, 22, 23].

Now I express the hyperfine coupling Hamiltonian (2) in terms of the excitonic representation. By omitting the $\hat{I}_z\hat{\sigma}_z$ term due to its irrelevance to any spin-flip process [24], and substituting into Eq. (2) the Schrödinger operators $\hat{\Psi}^\dagger(\mathbf{R}) = \chi(z)\sum_p\left(a^\dagger_p + b^\dagger_p\right)\psi^*_p(\mathbf{r})$ and $\hat{\Psi}(\mathbf{R}) = \left(\hat{\Psi}^\dagger\right)^\dagger$ instead of Ψ^* and Ψ [$\chi(z)$ is size-quantized wave function, $\mathbf{r} = (x,y)$], one finds

$$\hat{H}_{\rm hf} = \frac{v_0}{4\pi l_B^2\sqrt{N_\phi}}\times$$
$$\times\sum_{\mathbf{q}} f(q)\mathcal{Q}_{\mathbf{q}}\sum_n A_n|\chi(Z_n)|^2 e^{i\mathbf{q}\mathbf{R}_n}\hat{I}^{(n)}_- + {\rm h.c.}, \qquad (5)$$

where $f = e^{-q^2/4}[L_k(q^2/2)]$ if $\nu = 2k+1$ (L_k is the Laguerre polynomial), and $f = e^{-q^2/4}$ if $\nu < 1$.

A set of I_z spin numbers $\{M\} = (M_1, M_2, ...M_n, ...)$, where every M_n can assume one of the values $-3/2, -1/2, 1/2, 3/2$, completely determines the state of the nuclear system. The case where 2DEG electrons are in the state $|N\rangle$ or $|N-1;1;\mathbf{q}\rangle$ and nuclei in the state $\{M\}$ I symbolize as $|\{M\};N\rangle$ and $|\{M\};N-1;1;\mathbf{q}\rangle$, respectively. Application of the decreasing/increasing operator $\hat{I}^{(n)}_\mp$ to the former yields $\hat{I}^{(n)}_\mp|\{M\},N\rangle = \sqrt{\left(\frac{5}{2}\mp M_n\right)\left(\frac{3}{2}\pm M_n\right)}\,|\{M\}^\mp_n, N\rangle$, where $\{M\}^\mp_n = (M_1, M_2, ...M_n \mp 1, ...)$.

From this point the study of the relaxation rate becomes similar to that in Ref. [14]. The only appreciable difference is the presence of the nuclear component. The temperature is again assumed to be negligible. Being of the same order or even smaller than the uncertainty value determined by the external smooth disorder field [14] it is, in particular, well smaller than the Zeeman gap $\epsilon_{\rm Z}$. The initial state $|i\rangle$ is thus the Goldstone condensate containing N zero spin excitons: $|i\rangle = |\{M\};N\rangle$. This state is electronically and nuclearly degenerate. The GM breakdown is studied in terms of the transitions governed by the Fermi Golden Rule probability: $w_{if} = (2\pi/\hbar)|\mathcal{M}_{if}|^2\delta(E_f - E_i)$, where the final state $|f\rangle$ is the state where a part of the Zeeman energy has been converted into the nonzero spin exciton kinetic energy $\mathcal{E}_q$. Such a transition is the $2X_0 + \{M\} \to X_{\mathbf{q}^*} + \{M\}^-_n$

process, if calculated in the lowest order of the perturbation theory. The final state for this transition is $|f\rangle = |\{M\}_n^-; N-2; 1; \mathbf{q}^*\rangle$, where q^* is determined by the energy conservation equation $2\epsilon_Z = \epsilon_Z + \mathcal{E}(q^*)$, i.e. $q^* = \sqrt{2M_\mathrm{x}\epsilon_Z}$. When calculating the transition matrix element $\mathcal{M}_{if}(n, \mathbf{q}^*) = \langle f|\hat{H}_{hf}|i\rangle$, one may take into account that $q^* \ll 1$. So, the squared value is

$$|\mathcal{M}_{if}(n,\mathbf{q})|^2 = \frac{v_0^2 A_n^2 |\chi(Z_n)|^4}{(4\pi)^2 l_B^4 \mathcal{N}_\phi} \times$$
$$\times \left(\frac{5}{2} - M_n\right)\left(\frac{3}{2} + M_n\right) \frac{|\langle \mathbf{q}; 1; N-2|\mathcal{Q}_{-\mathbf{q}}|N\rangle|^2}{R(N)R(N-2;1;\mathbf{q})}, \quad (6)$$

where the notation $R(...)$ stands for the norm of the state $|...\rangle$. Now, at variance with the cited works [13, 14], the expectations entering Eq. (6) should be calculated not only for the integer QHF but for the fractional QHF too. The latter can be obtained within the so-called "single-mode approximation" [2, 3], namely:

$$\langle \mathbf{q}; 1; N-2|\mathcal{Q}_{-\mathbf{q}}|N\rangle = -\frac{\nu' N! (\nu' N_\phi - 2)!}{N_\phi^{N-1} (\nu' N_\phi - N)!} [1 + O(q^4)],$$

$$R(N) = \frac{N! (\nu' N_\phi)!}{N_\phi^N (\nu' N_\phi - N)!}, \quad \text{and}$$

$$R(N;1;\mathbf{q}) = \frac{\nu' N! (\nu' N_\phi - 2)!}{N_\phi^N (\nu' N_\phi - N - 2)!} \left[1 + O(q^4)\right]. \quad (7)$$

Here $\nu' = \nu$ if $\nu < 1$ or $\nu' = 1$ if $\nu = 2k+1$ ($k = 0, 1, 2, ...$). Formulas (7) are exact for odd-integer ν [25] (then the $\sim O(q^4)$ terms vanish), but for $\nu < 1$ they represent a result of the semi-phenomenological approach where the expectations are expressed in terms of the two-particle correlation function calculated for Laughlin's states [26].

Using Eqs. (6), (7), and assuming that the nuclei are unpolarized, I get the rate of the considered $S_z \to S_z + 1$ process:

$$-dN/dt = \frac{2\pi}{\hbar} \sum_{n,\mathbf{q}} |\mathcal{M}_{if}(n,\mathbf{q})|^2 \, \delta(q^2 l_B^2 / 2M_\mathrm{x} - \epsilon_Z) =$$
$$= \frac{N(N-1)}{\nu' N_\phi \tau_\mathrm{hf}} \quad (\text{for } \mathit{any}\ N \geq 1), \quad (8)$$

where

$$1/\tau_\mathrm{hf} = \frac{5 v_0 M_\mathrm{x} \left(A_\mathrm{Ga}^2 + A_\mathrm{As}^2\right)}{8\hbar l_B^2 d}. \quad (9)$$

Here d stands for a conventional width of the 2DEG: $1/d = \int |\chi(z)|^4 dz$. (This value certainly is not equal to the quantum well width d_QW, but constitutes a fraction of the latter, e.g.: $d/d_\mathrm{QW} \simeq 1/3$.) Formula (9) has been obtained for the case of unpolarized nuclei, i.e. $\overline{M_n} = 0$, and the correlation length of the nuclear momenta distribution is smaller than the magnetic length l_B, hence $\overline{M_n^2} = 5/4$ where the over-line means averaging over the volume $2\pi l_B^2 d$.

The elementary process just studied characterizes only the initial stage of the Goldstone condensate breakdown. Further physical picture of the relaxation follows absolutely the same scenario that was analyzed in Ref. [14]. When the Goldstone condensate is depleted, a "thermodynamic condensate" is developing. The latter is formed by spin waves with nonzero but negligibly small wave vectors, which are of the order of or smaller than the uncertainty value determined by smooth disorder. The number of nonzero excitons is equal to $|\Delta S|$, i.e. to deviation of the QHF total spin number S from its ground state value. $|\Delta S(t)|$ reaches the maximum value [still being well smaller than the simultaneous number $N(t)$], and in the final stage both condensates decay. By considering concentrations of the Goldstone and thermodynamic condensates – $n = N/\nu' N_\phi$ and $m = |\Delta S|/\nu' N_\phi$, one can find equations governing the relaxation,

$$\tau_\mathrm{hf} dn/dt = -2n^2 - 4mn \quad \text{and} \quad \tau_\mathrm{hf} dm/dt = n^2 - 2m^2. \quad (10)$$

These equations yield $n(t) = 1/[2n(0)t^2 + 2t + 1/n(0)]$ and $m = n(t)n(0)t$, where t is measured in τ_hf. I remind that, as in the work of [14], this result is analytical but still approximate – it should work well if $n(0) < 1/2$.

The only difference is thereby a change in Eqs. (10) from the characteristic relaxation time [14]

$$1/\tau_\mathrm{so} = 8\pi^2(\alpha^2 + \beta^2) M_\mathrm{x}^2 \epsilon_Z \overline{K}(q^*)/\hbar^3 \omega_c^2 l_B^4, \quad (11)$$

determined by the SO coupling and smooth random potential, to the HF coupling time τ_hf (9). (I keep notations of the paper [14]: α and β are the Rashba and Dresselhaus SO parameters, $\overline{K}(q)$ stands for the Fourier component of the smooth random potential correlator.) Comparing τ_so with τ_hf, one can note that they have opposite dependences on the magnetic field. If $K(r)$ is Gaussian, then $\overline{K}(q)$ is sharply decreasing with B: $\overline{K}(q^*) = (\Delta^2\Lambda^2/4\pi)\exp\left(-M_\mathrm{x}\epsilon_Z\Lambda^2/2l_B^2\right) \sim \exp(-\gamma M_\mathrm{x} B^2)$. ($\Delta$ and Λ stand for the amplitude and correlation length of the random potential, respectively.) Meanwhile, according to the above calculations, the HP rate $1/\tau_\mathrm{hp}$ is proportional to the squared local density $|\Psi(\boldsymbol{R}_n)|^4 \sim B^2$ and to the number of nuclei per electron $2\pi l_B^2 d/v_0 \sim 1/B$; therefore $1/\tau_\mathrm{hp} \sim M_\mathrm{x} B$.

More specific estimates of τ_hp and τ_so are required for appropriate comparison of both relaxation channels. The material parameters and characteristic parameters related to modern wide quantum-well structures could be, e.g., chosen as $v_0\left(A_\mathrm{Ga}^2 + A_\mathrm{As}^2\right) = 1.8 \cdot 10^{-4}\,\mathrm{meV^2 nm^3}$, $2(\alpha^2 + \beta^2)/(l_B \hbar\omega_c)^2 = 10^{-3}/B$, $\epsilon_Z = 0.0255B\,\mathrm{meV}$, $\Lambda = 60\,\mathrm{nm}$, $\Delta = 0.3\,\mathrm{meV}$, and $d = 8\,\mathrm{nm}$ (here B is assumed to be measured in Tesla; c.f. also estimates in Ref. [7]). However, estimate of the effective spin-exciton mass $M_{\rm x}$ strongly depends on the finite thickness structural formfactor. There are experimental data where $M_{\rm x}$ is found at comparatively low magnetic fields: (i) $1/M_{\rm x}\approx 1.2\,$meV at $B=2.27\,$T and $\nu=1$ in the 33 nm quantum well [22]; (ii) $1/M_{\rm x}\approx 1.51\,$meV at $B=2.69\,$T and $\nu=1$ in the 23 nm quantum well [23]; and (iii) $1/M_{\rm x}\approx 0.44\,$meV at $B=2.9\,$T and $\nu=1/3$ in the 25 nm quantum well [4]. For these fields characterized by the inequality $l_B>d$, the B-dependence should be approximately $1/M_{\rm x}\sim B^{1/2}$, but in the $l_B<d$ strong field regime the inverse mass grows much weaker with B. Based on these data, the semi-empirical analysis using characteristic GaAs/AlGaAs formfactors allows me to consider values $1/M_{\rm x}\simeq 2\,$meV at $\nu=1$ and $1/M_{\rm x}\simeq 0.67\,$meV at $\nu=1/3$ as the characteristic ones for the $10\,{\rm T}<B<20\,{\rm T}$ range. (Note that at a given field B the estimate $M_{\rm x}^{-1}|_{\nu<1}\simeq\nu'\cdot M_{\rm x}^{-1}|_{\nu=1}$ holds according to the semi-phenomenological theory [3].) Then, if substituting the above parameters into Eqs. (9) and (11), one obtains $\tau_{\rm hf}(B^*)=\tau_{\rm so}(B^*)$ at $B^*\approx 15\,$T if $\nu=1$, or at $B^*\approx 9.3\,$T if $\nu=1/3$. The characteristic relaxation time at these crossover points constitutes $\simeq 4\,\mu$s or $\simeq 2\,\mu$s respectively.

To conclude, I have reported on a new spin relaxation channel in the spin polarized strongly correlated 2DEG. The mechanism involves only the hyperfine coupling to GaAs nuclei, and no other interactions are required for this relaxation channel to be realized. The problem is solved by using the *excitonic representation* technique. Although the Goldstone mode relaxation in a QHF occurs by the scenario studied earlier [14], a crossover from the SO characteristic relaxation time (11) to the hyperfine coupling time (9) occurs in a strong magnetic field $B\gtrsim 10\,$T.

The work is supported by the RFBR.

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