

# Detecting non-Abelian statistics of Majorana fermions in quantum nanowire networks

Z.-Y. Xue

*Laboratory of Quantum Information Technology, School of Physics and Telecommunication Engineering,  
South China Normal University, Guangzhou 510006, China*

Submitted 17 June 2011

We propose a scheme in semiconducting quantum nanowires structure to demonstrate the non-Abelian statistics for Majorana fermions in terms of braid group. The Majorana fermions are localized at the endpoints of semiconducting wires, which are deposited on an  $s$ -wave superconductor. The non-Abelian nature of Majorana fermion is manifested by the fact that the output of the different applied orders of two operations, constructed by the braid group elements, are different. In particular, the difference can be unambiguously imprinted on the quantum states of a superconducting flux qubit.

Quantum statistics is a fundamental concept in physics which distinguishes fermions from bosons in three dimensions. For quasi-particles live in two dimensions, they may have two different classes of exotic statistics: Abelian or non-Abelian statistics. When one particle is exchanged in a counterclockwise manner with the nearest particle, the relation between the initial wave function  $\psi(\mathbf{r}_i, \mathbf{r}_j)$  with the final wave function  $\psi'(\mathbf{r}_i, \mathbf{r}_j)$  is given by  $\psi'(\mathbf{r}_i, \mathbf{r}_j) = M_{ij}\psi(\mathbf{r}_i, \mathbf{r}_j)$ , where  $\mathbf{r}_i$  is the position of the particle  $i$ . The particles are called Abelian anyons [1] with the statistics angle  $\theta$  if  $M_{ij} = \exp(i\theta)$  where  $\theta$  is not zero or  $\pi$  ( $\theta = 0$  and  $\pi$  correspond to bosons and fermions, respectively). Furthermore,  $M_{ij}$  can be the elements of a braid group, where two elements  $M_{ij}$  and  $M_{i'j'}$  may even be non-commutative, and quasi-particles with such features are called non-Abelian anyons [2].

Potential host systems for the exotic non-Abelian statistics including the  $\nu = 5/2$  quantum Hall state [3, 4], chiral  $p$ -wave superconductors [5], topological insulator superconductor [6] and semiconductor-superconductor structures [7–9]. With the potential applications in topological quantum computation, Majorana fermions (MF) with non-Abelian statistics have attracted strong renewed interests. MF are a kind of self-conjugate quasi-particles induced from a vortex excitation in  $p_x + ip_y$  superconductor. However, due to the instability of the  $p$ -wave superconducting states, its implementation still remains an experimental challenge. Therefore, setups with  $s$ -wave superconductor proximity effect [6–9], which is more stable, is more preferred. In principle, they should allow robust topological superconducting phase without unrealistic experimental conditions. More recently, it is recognized that topologically protected states may be most easily engineered in 1D-semiconducting wires deposited on an  $s$ -wave superconductor [10–12]: the end-

points of such wires support localized zero-energy MF. This setup provides the first realistic experimental setting for Kitaev's 1D topological superconducting state [13]: MF can be created, transported, fused and braided by applying locally tunable gates to the wire. However, detecting MF in such setup is a challenge task because they hold neutral charge. Recently, it has been shown that combine two MF into a single Dirac fermion allows the neutral quasi-particles to be probed with charge transport [14–17]. But, they suffer from the same impediment: Abrikosov vortices are so massive objects that behave classically. Fortunately, alternatives using Josephson vortices are proposed by introducing a controllable superconducting flux qubit [18–20], which enable one to detect unambiguously the states of MF in this 1D-scenario.

Recently, Zhu et al. [21] proposed a scheme to directly test the quantum statistics of the braid group for MF in cold atoms scenario. Confirming this aspect of MF is not only of significant important in its own, but also the crucial and first step towards the realization of topological quantum computation. However, simulate exotic statistics of MF in a macroscopic material is another story. Here, we propose such an alternative scheme to detect the non-Abelian statistics of the MF in terms of braid group in semiconducting quantum nanowires structure. The non-Abelian nature is manifested by the fact that the output of the different applied orders of two operations are different, which is different from previous schemes based on charge transport [14–18]. The operations are constructed by the braid group elements of MF. Furthermore, the two different final states can be distinguished by measuring the states of MF by the superconducting flux qubit.

The setup we consider is shown in Fig. 1, which is a spin-orbit coupled semiconducting wire deposited on

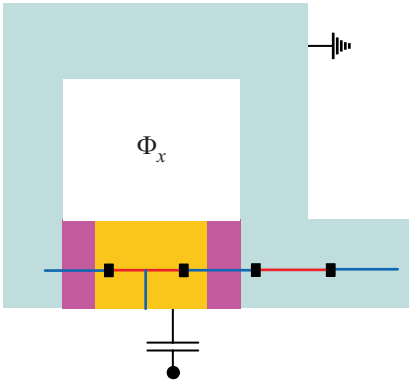


Fig. 1. (Color online) Superconducting flux qubit with two Josephson junctions (pink) and an enclosing magnetic flux  $\Phi_x$ . MF (black) are induced at the interface between a topologically trivial (blue) and a topologically nontrivial (red) section of a quantum nanowire. Gate electrodes (not shown) can be used to move the MF along the wire

an  $s$ -wave superconductor. Applying a magnetic field perpendicular to the superconductor surface, the Hamiltonian describing such a wire is [10]

$$H = \int \left[ \psi_x^\dagger \left( -\frac{\hbar^2 \partial_x^2}{2m} - \mu - i\hbar u \hat{e} \cdot \boldsymbol{\sigma} \partial_x + V_B \sigma_z \right) \psi_x + (|\Delta| e^{i\varphi} \psi_{\downarrow x} \psi_{\uparrow x} + \text{h.c.}) \right] dx, \quad (1)$$

where  $\psi_{\alpha x}$  corresponds to electrons with spin  $\alpha$ , effective mass  $m$ , and chemical potential  $\mu$ ; the third term denotes spin-orbit coupling with  $u$  the strength, and  $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$  is the vector of Pauli matrices; the fourth term represents the energy shift due to the magnetic field; and the terms in the second line are the spin-singlet pairing from the  $s$ -wave superconductor via proximity effect.

In the setup, the magnetic field is weak, and the superconductor of the flux qubit is the conventional  $s$ -wave superconductor. The interplay of Zeeman effect, spin-orbit coupling, and the proximity to an  $s$ -wave superconductor drive the wire into a chiral  $p$ -wave superconducting state [10–12], providing that the wire is long compared to the superconducting coherence length ( $\xi \simeq 40$  nm for the superconducting substrate being Nb). Specifically, when  $|\Delta| < |V_B|$ ,  $\mu$  lies inside of the zero-mode energy gap, the wire is in the Kitaev's topological phase [13], which supports Majorana modes; otherwise the system is an ordinary superconductor (topological trivial). The zero-mode excitation gap and  $\mu$  dependence is [12]  $E_0 = \left| |V_B| - \sqrt{|\Delta|^2 + \mu^2} \right|$ . For  $|\mu| < \mu_c = \sqrt{V_B^2 - |\Delta|^2}$  the topological phase with end Majoranas emerges, or a topologically trivial phase. Thus,

applying a gate voltage uniformly deleted allows one to create or remove the MF. To avoid gap closure, A “keyboard” of local tunable gate electrodes to the wire [12] is used to control whether a region of the wire is topological or not. For InAs quantum nanowire, assuming  $|V_B| \sim 2|\Delta|$  and  $\hbar u \sim 0.1$  eVÅ, the gap for a  $0.1 \mu\text{m}$  wide gate is of order 1 K [12]. It is note that heavy-element wires and/or narrower gates could generate even larger gaps. Therefore, MF are induced at the interface between topologically trivial and nontrivial sections of the quantum nanowire.

For a pair of MF, they can be combined to form a complex fermionic states  $c = \gamma_2 + i\gamma_1$ , which can be occupied  $|1\rangle$  or unoccupied  $|0\rangle$ , differ in fermion parity, and therefore 2-fold degenerated. A winding of one MF around another is associated with a unitary transformation in the subspace of degenerated ground states. For  $2N$  MF, such unitary transformations form a set named braid group, which is generated by elementary interchanges of neighboring MF [22].

For our purpose to verify the no-Abelian nature of MF, four MF is enough. They combine into two complex fermions  $c_1$  and  $c_2$  and the ground state has degeneracy four, but the Hamiltonian (1) conserves parity of the fermion number, and the even-number subspace is decoupled from the odd-number subspace. At low temperature (significantly below the zero-mode gap), the initial state is typically a vacuum state  $|0\rangle$ , and in the even-number subspace  $\{|0\rangle|0\rangle, |1\rangle|1\rangle\}$ , only two of the three generators are independent:

$$\tau_1 = \tau_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 - i & 0 \\ 0 & 1 + i \end{pmatrix}, \quad (2)$$

$$\tau_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}. \quad (3)$$

From these, two composite braiding operations [21]

$$A = \tau_1 \tau_2 \tau_1^{-1}, \quad B = \tau_1^{-1} \tau_2 \tau_1^{-1} \quad (4)$$

can be constructed with the property  $AB = i\sigma_z$ ,  $BA = -i\sigma_x$ , i.e.,  $AB \neq BA$ , manifested the non-Abelian nature of MF, where  $\tau_i$  and  $\tau_i^{-1}$  denote counter-clockwise and clockwise interchange of MF  $i$  and  $i + 1$ , respectively. Specifically, with the initial state  $|0\rangle|0\rangle$ ,  $AB$  and  $BA$  yield two orthogonal output states  $|0\rangle|0\rangle$  and  $|1\rangle|1\rangle$ , respectively. We on purposely chose the two outputs to be orthogonal with each other so that the difference between the two can be detected unambiguously.

However, to implement braiding of MF in a 1D-structure is impossible. Therefore, in order to exchange

the MF for braiding operations, one can use a second wire to form a  $T$ -junction or more efficiently with the “railroad track” geometry [12]. The two red sections in Fig. 1 is tune into the topological phase while the blue sections are not. In this way, we induce two pairs of MF with the left pair (labeled as 1 and 2) residents in the superconducting island of the flux qubit; the right pair of MF is labeled as 3 and 4. Another wire, perpendicular to the wire holds MF, is introduced to enable interchange of neighboring MF. In our setup, to demonstrate the non-Abelian nature of MF, i.e., to implement braiding operations in Eq. (4), we need to counterclockwise interchange of MF 1, 2 and 2, 3 for  $\tau_1$  and  $\tau_2$ , respectively. Counterclockwise interchange of MF can be achieved along the line as proposed in Ref. [12]. For example,  $\tau_1$  can be implemented as following: transports MF 1 downward by driving the vertical wire into a topological phase; transports MF 2 leftward in a similar fashion; and finally transports MF 1 up and to the right. Similarly,  $\tau_1^{-1}$  can be implemented as following: transports MF 2 downward, transports MF 1 rightward and finally transports MF 2 up and to the left.

We now turn to detect the two orthogonal output states  $|0\rangle_L|0\rangle_R$  and  $|1\rangle_L|1\rangle_R$ . As the output states of the two pairs of MF are the same, detecting one of them can then fulfil the purpose of distinguishing the output states. The two states  $|0\rangle$  and  $|1\rangle$  are distinguished by the parity of the number  $n_p$  of particles in the island. Therefore, they can be distinguished by  $n_p$ . As  $|0\rangle_L|0\rangle_R$  and  $|1\rangle_L|1\rangle_R$  have the same parity, we need to remove one pair of MF out of the measurement circle, otherwise they will also contribute to the measurement results and make the detection impossible. Without loss of generality, we choose to detect the left pair of MF (MF 1 and 2) while move MF 3 and 4 along the wire out of the flux qubit circuit, as shown in Fig. 1, by local tunable gates. Note that the wire is not interrupted by the junctions providing that the junctions’ thickness is much smaller than  $\xi$ .

To measure the parity of  $n_p$ , we make use of the suppression of macroscopic quantum tunneling by the Aharonov-Casher effect [23]: a vortex encircling a superconducting island picks up a phase increment  $\phi = \pi q/e$  determined by the total charge  $q$  coupled capacitively to the superconductor. For our case, the two Josephson junctions have the same Josephson coupling energy  $E_J$ . The flux  $\Phi$  in the qubit is related to the phase differences across the junctions,  $\varphi_1$  and  $\varphi_2$ , by  $2\pi\Phi/\Phi_0 = \varphi_1 + \varphi_2$ ; the phase  $\theta = (\varphi_1 - \varphi_2)/2$  is conjugate to the number of excess Cooper pairs of the superconducting island as  $[\theta, n] = i$ . The potential energy is plotted in Fig. 2 for the external magnetic  $\Phi_x = \Phi_0/2$  and  $E_J = 0.507\Phi_0^2/(2L)$

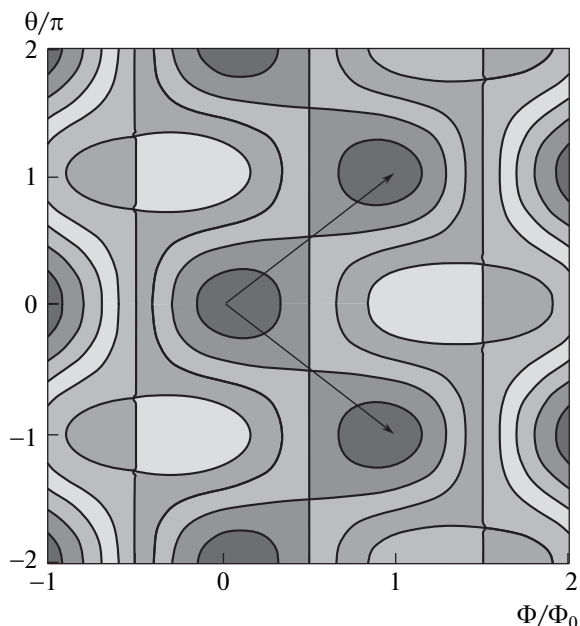


Fig. 2. Potential of the symmetric superconducting flux qubit with  $\Phi_x = \Phi_0/2$  and  $E_J = 0.507\Phi_0^2/(2L)$ . The energy minima are connected by two tunneling paths indicated by the red arrows

with  $L$  being the inductance of the flux qubit, which is strictly periodic in the  $\theta$  direction. Therefore, neighboring minima are always separated by  $\delta\theta = \pm\pi$  (as indicated by red arrows in Fig. 2), i.e., the energy minima are connected by two tunneling paths with same amplitude, which amounts to the circulation of a Josephson vortex around the superconducting island. The interference produces an oscillatory tunnel splitting of the two levels of the flux qubit

$$\Delta = \Delta_0 \cos(\phi/2), \quad (5)$$

where  $\Delta_0$  is the tunnel splitting associated with one path. Therefore, if  $q$  is an odd (even) multiple of the electron charge  $e$ , the two tunneling paths interfere destructively (constructive), so the tunnel splitting is minimum (maximal).

As we only need to distinguish maximal from minimal tunnel splitting, the flux qubit does not need to have a large quality factor. In addition,  $\Delta_0 \simeq 100 \mu\text{eV} \simeq 1 \text{ K}$  for parameters in typical experiments of flux qubits [23], which should be readily observable by microwave absorption. To make sure the total charge is solely comes from  $|1\rangle_L$ , one would first calibrate the charge on the gate capacitor to zero, by maximizing the tunnel splitting in the absence of vortices in the island. Meanwhile, the read-out is nondestructive, which is necessary for the realization of a two-qubit CNOT gate [18]. More-

over, the read out is insensitive to sub-gap excitations in the superconductor (they do not change the fermion parity).

In summary, we have proposed a scheme in semiconductor-superconductor structure to demonstrate the non-Abelian statistics for MF in terms of braid group. The non-Abelian nature is manifested by the fact that the output of the different applied orders of two operations, constructed by the braid group elements, are different. Meanwhile, the different final state of MF can be unambiguously detected by a superconducting flux qubit.

This work was supported by the NSFC (# 11004065), the NSF of Guangdong Province (# 10451063101006312), and the Startup Foundation of SCNU (# S53005).

- 
1. F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
  2. G. Moore and N. Read, Nucl. Phys. B **360**, 362 (1991).
  3. N. Read and D. Green, Phys. Rev. B **61**, 10267 (2000).
  4. S. Das Sarma, M. Freedman, and C. Nayak, Phys. Rev. Lett. **94**, 166802 (2005).
  5. S. Das Sarma, C. Nayak, and S. Tewari, Phys. Rev. B **73**, 220502(R) (2006).
  6. L. Fu and C.L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
  7. J. D. Sau, R. M. Lutchyn, S. Tewari, and S. Das Sarma, Phys. Rev. Lett. **104**, 040502 (2010).
  8. J. Alicea, Phys. Rev. B. **81**, 125318 (2010).
  9. P. Bonderson, S. Das Sarma, M. Freedman, and C. Nayak, arXiv:1003.2856.
  10. R. M. Lutchyn, J. D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010).
  11. Y. Oreg, G. Refael, and F. von Oppen, Phys. Rev. Lett. **105**, 177002 (2010).
  12. J. Alicea, Y. Oreg, G. Refael et al., Nature Phys. **7**, 412 (2011).
  13. A. Y. Kitaev, Physics-Uspekhi **44**, 131 (2001).
  14. L. Fu and C.L. Kane, Phys. Rev. Lett. **102**, 216403 (2009).
  15. A. R. Akhmerov, J. Nilsson, and C.W.J. Beenakker, Phys. Rev. Lett. **102**, 216404 (2009).
  16. J. D. Sau, S. Tewari, and S. Das Sarma, arXiv:1004.4702.
  17. E. Grosfeld, B. Seradjeh, and S. Vishveshwara, Phys. Rev. B **83**, 104513 (2011).
  18. F. Hassler, A. R. Akhmerov, C.-Y. Hou, and C. W. J. Beenakker, New J. Phys. **12**, 125002 (2010).
  19. L. Jiang, C. L. Kane, and J. Preskill, Phys. Rev. Lett. **106**, 130504 (2011).
  20. P. Bonderson and R. M. Lutchyn, Phys. Rev. Lett. **106**, 130505 (2011).
  21. S.-L. Zhu, L.-B. Shao, Z. D. Wang, and L.-M. Duan, Phys. Rev. Lett. **106**, 100404 (2011).
  22. D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
  23. J. R. Friedman and D. V. Averin, Phys. Rev. Lett. **88**, 050403 (2002).