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Analytical five-loop expressions for the renormalization group QED β-function in different renormalization schemes

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We obtain analytical five-loop results for the renormalization group β-function of Quantum Electrodynamics with the single lepton in different renormalization schemes. The theoretical consequences of the results obtained are discussed.

The concept of the β-function, which depends on the choice of the renormalization scheme, is the cornerstone of the Quantum Field Theory renormalization group approach, developed in the works of Refs. [1–3]. In QED the study of the perturbative expansion of the β-function is of special interest. Indeed, it governs the energy-dependence of the constant $\alpha = e^2/(4\pi)$, which defines the coupling of photons with leptons. In this work we will obtain the five-loop analytical expressions for the renormalization group QED β-function of the electron, neglecting the contributions of leptons with higher masses, namely the contributions of muons and tau-leptons.

We will start with the expression for the β-function in the variant of the minimal subtraction scheme [4], namely the $\overline{MS}$-scheme [5]:

$$\beta_{\overline{MS}}(\bar{\alpha}) = \mu^2 \frac{\partial(\bar{\alpha}/\pi)}{\partial \mu^2} = \sum_{i\geq 1} \bar{\beta}_i \left(\frac{\bar{\alpha}}{\pi}\right)^{i+1}, \qquad (1)$$

where $\bar{\alpha}$ is the renormalized $\overline{MS}$-scheme QED coupling constant and μ^2 is the $\overline{MS}$-scale parameter. At the three -loop level the scheme-dependent coefficient $\bar{\beta}_3$ was independently evaluated analytically in [6, 7]. The four-loop coefficient $\bar{\beta}_4$ was obtained as the result of the project, started in Ref. [8] and completed in Ref. [9]. The QED result of Ref. [9] was confirmed after taking the QED limit of the analytically evaluated in Ref. [10] 4-loop correction to the $\overline{MS}$-scheme β-function of the $SU(N_c)$ colour gauge model.

To get the five-loop expression for the coefficient $\bar{\beta}_5$ we use the derived in Ref. [11] renormalization-group expression, which has the following form:

$$\bar{\beta}_5 = -\bar{b}_5 - 3\bar{b}_1\bar{a}_4 - 2\bar{b}_2\bar{a}_3 - \bar{b}_3\bar{a}_2 - \bar{b}_1\bar{a}_2^2, \qquad (2)$$

where $\bar{a}_l$ and $\bar{b}_l$ enter into the expressions for the l-loop contributions to the photon vacuum polarization functions with $1 \leq l \leq 5$. These contributions are defined as

$$\bar{\Pi}_l(x) = [\bar{a}_l + \bar{b}_l \ln(x) + \bar{c}_l \ln^2(x) + + \bar{d}_l \ln^3(x) + \bar{e}_l \ln^4(x)] \left(\frac{\bar{\alpha}}{\pi}\right)^{l-1}, \qquad (3)$$

where $x = Q^2/\mu^2$ and Q^2 is the Euclidean momentum transfer. It is possible to show, that for $l \leq 2$ $\bar{c}_l = 0$, $\bar{d}_l = 0$, $\bar{e}_l = 0$, for $l \leq 3$ $\bar{d}_l = 0$ and $\bar{e}_l = 0$, while for $l = 4$ $\bar{e}_4 = 0$. In general $\bar{c}_l$, $\bar{d}_l$, $\bar{e}_l$ are expressed through the products of lower order coefficients $\bar{b}_i$ via the corresponding renormalization group relations (see Refs. [12, 11]). For $1 \leq l \leq 3$ the coefficients $\bar{a}_i$ and $\bar{b}_i$ are defined by the results of Ref. [12] and read $\bar{a}_1 = 5/9$, $\bar{a}_2 = 55/48 - \zeta_3$, $\bar{a}_3 = -1247/648 - (35/72)\zeta_3 + (5/2)\zeta_5$, $\bar{b}_1 = -1/3$, $\bar{b}_2 = -1/4$, $\bar{b}_3 = 47/96 - \zeta_3/3$, while the expression for $\bar{a}_4 = 1075825/373248 - (13/96)\zeta_4 + (13051/2592)\zeta_3 - (5/3)\zeta_3^2 + (45/32)\zeta_5 - (35/4)\zeta_7$ was evaluated in Ref. [13]. In order to get $\bar{\beta}_5$ from Eq. (2) we should fix the analytical expression for $\bar{b}_5$ in the r.h.s. of Eq. (2). It is composed of the sum of three contributions $\bar{b}_5 = \bar{b}_5[nlbl] + \bar{b}_5[3, lbl] + \bar{b}_5[2, lbl]$. The first term in $\bar{b}_5$ is fixed by the leading logarithmic term of the sum of five-loop photon vacuum polarization graphs, which contain the electron loop with two external vertexes. The second term $\bar{b}_5[3, lbl]$ comes from the logarithmic contribution to five-loop photon propagator diagrams with two one-loop light-by-light scattering type sub-graphs, connected by a photon line with an electron loop insertion and two undressed photon propagators. The third term $\bar{b}_5[2, lbl]$ arises from the logarithmic contribution to five-loop photon propagator diagrams with one-loop and two-loop light-by-light scattering type sub-graphs, connected by three undressed photon propagators.

The first contribution into $\bar{b}_5$ can be defined from the QED limit of the result of Ref. [14] with one active lepton and reads

$$\bar{b}_5[nlbl] = -\frac{409367}{1492992} + \frac{15535}{2592}\zeta_3 - \zeta_3^2 + \frac{25}{12}\zeta_5 - \frac{35}{4}\zeta_7. \qquad (4)$$

The term $\bar{b}_5[3, lbl]$ is fixed by us from the QED limit of the depending on the number of fermions analytical expression for the order α_s^4 singlet QCD contribution to the cross-section for electron-positron annihilation into hadrons, published in Ref. [15]. It has the following form

$$\bar{b}_5[3, lbl] = \frac{149}{108} - \frac{13}{6}\zeta_3 - \frac{2}{3}\zeta_3^2 + \frac{5}{3}\zeta_5. \qquad (5)$$

The term $\bar{b}_5[2, lbl]$ is obtained from the QED limit of the $C_F\alpha_s^4$ singlet QCD contribution to the cross-section for electron-positron annihilation into hadrons, presented in the subsequent works of Refs. [16, 17], with C_F being the quadratic Casimir operator of the $SU(N_c)$ colour gauge group. This term is

$$\bar{b}_5[2, lbl] = \frac{13}{12} + \frac{4}{3}\zeta_3 - \frac{10}{3}\zeta_5. \qquad (6)$$

Substituting the results from Eqs. (4), (5), and (6) into the r.h.s. of Eq. (2) we get the analytical expression for the five-loop approximation of the $\overline{MS}$-scheme QED β-function with a single lepton:

$$\beta_{\overline{MS}}(\bar{\alpha}) = \mu^2\frac{\partial(\bar{\alpha}/\pi)}{\partial\mu^2} = \sum_{i\geq 1}\bar{\beta}_i\left(\frac{\bar{\alpha}}{\pi}\right)^{i+1} = = \frac{1}{3}\left(\frac{\bar{\alpha}}{\pi}\right)^2 + \frac{1}{4}\left(\frac{\bar{\alpha}}{\pi}\right)^3 - - \frac{31}{288}\left(\frac{\bar{\alpha}}{\pi}\right)^4 - \left(\frac{2785}{31104} + \frac{13}{36}\zeta_3\right)\left(\frac{\bar{\alpha}}{\pi}\right)^5 + + \left(-\frac{195067}{497664} - \frac{13}{96}\zeta_4 - \frac{25}{96}\zeta_3 + \frac{215}{96}\zeta_5\right)\left(\frac{\bar{\alpha}}{\pi}\right)^6 + O(\bar{\alpha}^7), \qquad (7)$$

which contain the contributions of the Riemann ζ-functions, defined as $\zeta_k = \sum_{n=1}^{\infty}(1/n)^k$. Let us remind that scheme-dependent coefficients of the β-function do not depend on the concrete realization of the minimal subtraction scheme (see, e.g., [7]). Notice the appearance of the ζ_4-term in the expression for $\bar{\beta}_5$, which did not manifest itself in the lower order coefficients. This feature was already observed in Ref. [18] as the result of analytical calculations of the cubic in the number of leptons five-loop terms of the the QED $\beta_{\overline{MS}}$ -function, which did not contain the cubic in the number of leptons light-by-light-type terms. Comparing our result of Eq. (7) with the expression from Ref. [18], we conclude that the addition of the 5-loop light-by-light-type contributions changes the coefficient and sign of the ζ_4-contribution in the overall expression for $\bar{\beta}_5$ given in Eq. (7). This happens due to taking into account in the second term of Eq. (2) the light-by-light-type contribution into the constant term $\bar{a}_4$. Another intriguing observation is the cancellation in Eq. (7) of the ζ_7 and ζ_3^2 transcendentalities, which contribute to the first term in Eq. (2), namely the $\bar{b}_5[nlbl]$-term (see Eq. (4)).

Let us now transform Eq. (2) from the $\overline{MS}$- to the on-shell scheme using the following equation

$$\beta_{OS}(\alpha) = \sum_{i\geq 1}\beta_i\left(\frac{\alpha}{\pi}\right)^{i+1} = \beta_{\overline{MS}}[\bar{\alpha}(\alpha)]/\frac{\partial\bar{\alpha}(\alpha)}{\partial\alpha}, \qquad (8)$$

where $\mu^2 = m^2$, m is the electron pole mass, α is the QED coupling constant, defined in the on-shell scheme and

$$\bar{\alpha}(\alpha) = \alpha\left[1 + g_2\left(\frac{\alpha}{\pi}\right)^2 + g_3\left(\frac{\alpha}{\pi}\right)^3 + g_4\left(\frac{\alpha}{\pi}\right)^4 + O(\alpha^5)\right]. \qquad (9)$$

The coefficients $g_2 = 15/16$, $g_3 = -4867/5184 + (23/72)\pi^2 - (1/3)\pi^2\ln 2 + (11/96)\zeta_3$ were evaluated in Ref. [19], while

$$g_4 = 14327767/9331200 + (8791/3240)\pi^2 + (204631/259200)\pi^4 - (175949/4800)\zeta_3 + (1/24)\pi^2\zeta_3 + (9887/480)\zeta_5 - (595/108)\pi^2\ln 2 - (106/675)\pi^4\ln 2 + (6121/2160)\pi^2\ln^2 2 - (32/135)\pi^2\ln^3 2 - (6121/2160)\ln^4 2 + (32/225)\ln^5 2 - (6121/90)a_4 - (256/15)a_5$$

with a_4 and a_5 defined as $a_k = \mathrm{Li}_k[1/2] = \sum_{n=1}^{\infty}(1/2n)^k$ was obtained in Ref. [13].

Using these results in the transformation relations of Eqs. (9) and (8) we get

$$\beta_{OS}(\alpha) = m^2\frac{\partial(\alpha/\pi)}{\partial m^2} = \sum_{i\geq 1}\beta_i\left(\frac{\alpha}{\pi}\right)^{i+1} = = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 + \frac{1}{4}\left(\frac{\alpha}{\pi}\right)^3 - \frac{121}{288}\left(\frac{\alpha}{\pi}\right)^4 + + \left(\frac{5561}{10368} - \frac{23}{18}\zeta_2 + \frac{4}{3}\zeta_2\ln 2 - \frac{7}{16}\zeta_3\right)\left(\frac{\alpha}{\pi}\right)^5 + + \left[-\frac{23206993}{37324800} + \frac{6121}{2160}\ln^4 2 - \frac{32}{225}\ln^5 2 - - \frac{205021}{259200}\pi^4 + \frac{106}{675}\pi^4\ln 2 + \frac{6121}{90}\mathrm{Li}_4(1/2) + + \frac{256}{15}\mathrm{Li}_5(1/2) - \frac{36199}{12960}\pi^2 + \frac{151}{27}\pi^2\ln 2 - - \frac{6121}{2160}\pi^2\ln^2 2 + \frac{32}{135}\pi^2\ln^3 2 - - \frac{1}{24}\pi^2\zeta_3 + \frac{349123}{9600}\zeta_3 - \frac{2203}{120}\zeta_5\right]\left(\frac{\alpha}{\pi}\right)^6 + O(\alpha^7). \qquad (10)$$

The third coefficient coincides with the one, originally calculated in Ref. [20]. The agreement between analytical results for $\bar{\beta}_3$ and β_3-coefficients was first demonstrated in Ref. [7]. The expression for the four-loop coefficient β_4 is in agreement with the result of Ref. [19].

The five-loop coefficient β_5 is new. Note, that both β_4- and β_5-terms contain typical to the on-shell renormalization procedure contributions, which are proportional to ln2 and $\zeta_2 = \pi^2/6$. However, at present we are unable to rewrite the proportional to π^2 contributions into β_5 through the ζ-functions of even arguments. Indeed, the π^4-contributions to g_4 may be decomposed into the sum of ζ_4- and ζ_2^2-terms with unknown to us coefficients.

In order to get the five-loop expression for the QED Gell-Mann–Low $\Psi(\tilde{\alpha})$-function, which coincides with the QED β-function in the momentum (MOM) subtractions scheme (for the detailed explanation of this statement at the four-loop level see Refs. [8, 9]) we supplement the general transformation relation between the $\beta_{OS}(\alpha)$-function and the Ψ-function, derived in Ref. [11] with the explicit results for the on-shell scheme analogs of the coefficients $\bar{a}_3$ and $\bar{a}_4$ in Eq. (3), which are known from the results of Refs. [19, 13] respectively. The obtained result reads

$$\Psi(\tilde{\alpha}) = \mu^2 \frac{\partial(\tilde{\alpha}/\pi)}{\partial\mu^2} = \sum_{i\geq 1} \Psi_i \left(\frac{\tilde{\alpha}}{\pi}\right)^{i+1} = \frac{1}{3}\left(\frac{\tilde{\alpha}}{\pi}\right)^2 + \frac{1}{4}\left(\frac{\tilde{\alpha}}{\pi}\right)^3 + \left(-\frac{101}{288} + \frac{1}{3}\zeta_3\right)\left(\frac{\tilde{\alpha}}{\pi}\right)^4 + \left(\frac{93}{128} + \frac{1}{3}\zeta_3 - \frac{5}{3}\zeta_5\right)\left(\frac{\tilde{\alpha}}{\pi}\right)^5 + \left(-\frac{122387}{55296} - \frac{79}{24}\zeta_3 + \zeta_3^2 - \frac{185}{72}\zeta_5 + \frac{35}{4}\zeta_7\right) \times \left(\frac{\tilde{\alpha}}{\pi}\right)^6 + O(\tilde{\alpha}^7). \quad (11)$$

We checked that the identical result is obtained from the five-loop expression for the $\beta_{\overline{MS}}(\bar{\alpha})$-function after transforming it into the MOM-scheme. The expressions for Ψ_3 and Ψ_4 coincide with the results, originally obtained in Refs. [21, 9]. The expression for Ψ_5 is new. On the contrary to the five-loop $\overline{MS}$- and on-shell scheme coefficients $\bar{\beta}_5$ and β_5 it does not contain ζ-functions of even arguments. However, the contributions of ζ_7- and ζ_3^2-terms manifest themselves in Ψ_5 only. They are related to the similar scheme-independent [13] contributions into the non-logarithmic four-loop coefficients $\bar{a}_4$ and a_4 of the renormalized photon vacuum polarization function in the $\overline{MS}$- and on-shell scheme, as given in Ref. [13].

For the completeness we present the five-loop expression for the perturbative quenched QED contribution to the QED β-functions. It was originally obtained in Ref. [22] and published later in Ref. [14] after additional theoretical cross-checks proposed in Ref. [23]. The result reads

$$F_1(\alpha_*) = \frac{1}{3}\left(\frac{\alpha_*}{\pi}\right) + \frac{1}{4}\left(\frac{\alpha_*}{\pi}\right)^2 - \frac{1}{32}\left(\frac{\alpha_*}{\pi}\right)^3 - \frac{23}{128}\left(\frac{\alpha_*}{\pi}\right)^4 + \left(\frac{4157}{6144} + \frac{1}{8}\zeta_3\right)\left(\frac{\alpha_*}{\pi}\right)^5 + O(\alpha_*^6), \quad (12)$$

where α_* is the corresponding expansion parameter, while the coefficients of F_1-function do not depend from the renormalization scheme.

At the three- and four-loop level the related expressions were obtained in Refs. [24, 9]. The 4-loop result was independently confirmed later on in the work of Ref. [25].

In the numerical form the five-loop perturbative series we are interested in read

$$\beta_{\overline{MS}}(\bar{\alpha}) = 0.3333\left(\frac{\bar{\alpha}}{\pi}\right)^2 + 0.25\left(\frac{\bar{\alpha}}{\pi}\right)^3 - 0.1076\left(\frac{\bar{\alpha}}{\pi}\right)^4 - 0.5236\left(\frac{\bar{\alpha}}{\pi}\right)^5 + 1.471\left(\frac{\bar{\alpha}}{\pi}\right)^6, \quad (13)$$

$$\beta_{OS}(\alpha) = 0.3333\left(\frac{\alpha}{\pi}\right)^2 + 0.25\left(\frac{\alpha}{\pi}\right)^3 - 0.4201\left(\frac{\alpha}{\pi}\right)^4 - 0.5712\left(\frac{\alpha}{\pi}\right)^5 - 0.3462\left(\frac{\alpha}{\pi}\right)^6, \quad (14)$$

$$\Psi(\tilde{\alpha}) = 0.3333\left(\frac{\tilde{\alpha}}{\pi}\right)^2 + 0.25\left(\frac{\tilde{\alpha}}{\pi}\right)^3 + 0.04999\left(\frac{\tilde{\alpha}}{\pi}\right)^4 - 0.6010\left(\frac{\tilde{\alpha}}{\pi}\right)^5 + 1.434\left(\frac{\tilde{\alpha}}{\pi}\right)^6, \quad (15)$$

$$F_1(\alpha_*) = 0.3333\left(\frac{\alpha_*}{\pi}\right) + 0.25\left(\frac{\alpha_*}{\pi}\right)^2 - 0.03125\left(\frac{\alpha_*}{\pi}\right)^3 - 0.1797\left(\frac{\alpha_*}{\pi}\right)^4 + 0.8268\left(\frac{\alpha_*}{\pi}\right)^5. \quad (16)$$

Let us discuss the structure of these perturbative series. From the theoretical arguments, presented in the work of Ref. [26] one may expect that the β-functions are expanded into the sign-alternating asymptotic perturbative series with fast growing coefficients. And indeed, this feature is true in the case of the β-function of the $g\phi^4$-theory, which is known in the $\overline{MS}$-scheme up to the five-loop [27, 28]. In the case of QED the asymptotic estimates of Refs. [29, 30], analogous to Lipatov's ones for the $g\phi^4$-theory [26], indicate that the asymptotic structure of QED perturbative series is more complicated than in the $g\phi^4$-theory. Indeed, the asymptotic of Refs. [29, 30] were obtained only for the gauge-invariant subclasses of diagrams with fixed number of fermion loops. Moreover, the indication of the sign-alternating factorial growth of the perturbative coefficients of the F_1-function, given in Ref. [29], does not agree with the concrete behavior of the five-loop perturbative series, presented in Eq. (16). Besides, as it is discussed in Refs. [29, 30], in the case of complete QED the strong cancellations between coefficients of sub-sets of diagrams with different fixed numbers of fermion loops is expected. This effect may manifest itself in the differences of sign structures of the five-loop approximations for $\beta_{\overline{MS}}(\tilde{\alpha})$, $\beta_{OS}(\alpha)$, and $\Psi(\tilde{\alpha})$ (compare Eq. (13) with Eqs. (14) and (15)).

It is also interesting to note, that taking into account the calculated by us five-loop correction to $\Psi(\tilde{\alpha})$ confirms the confidence in the validity of the criterion $O \leq \Psi(\tilde{\alpha}) < \tilde{\alpha}/\pi$, derived by Schwinger [31] and Krasnikov [32] (see Ref. [33] as well). Note also, that the theoretical analysis of the behavior of the perturbative series for the Gell-Mann–Low function $\Psi(\tilde{\alpha})$, preformed in Ref. [34], which at large $\tilde{\alpha}$ indicates the validity of its linear behavior, supports the mentioned above identity, derived in Refs. [31–33].

However, we think that the the similar linear behavior, obtained in Ref. [35] for the QED β-function in the on-shell scheme for the case when the expansion parameter is going to infinity should be reconsidered. Indeed, analyzing the behavior of the perturbative series for the $\beta_{OS}(\alpha)$-function from Eq. (14) at the three-, four-, and five-loop levels, we observe the appearances of the rigorously speaking unphysical ultraviolet fixed points at $\alpha/\pi \approx 1.2$, $\alpha/\pi \approx 0.8$, and $\alpha/\pi \approx 0.7$ respectively. The appearances of these zeros may affect the exact asymptotic behavior of the QED β-function in the on-shell scheme, considered in Ref. [35].

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After the acceptance of this work for publication the analytical expression for the 5-loop QED corrections to the $\beta_{\overline{AS}}(\alpha)$ and Ψ-function with N-numbers of identical charges leptons became known [36].

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