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Asymptotics and zeros of the imaginary part of the elastic scattering amplitude

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The s-channel unitarity condition for the imaginary part of the hadronic elastic scattering amplitude outside the diffraction peak is studied within different assumptions about the behavior of its real part. The integral equation for the imaginary part is derived with the asymptotical expression for the real part inserted in the unitarity condition. The conclusions about the asymptotical approach to the black disk limit and possible zeros of the imaginary part of the amplitude are obtained. Their relation to the present day experiments is discussed.

The properties of high energy elastic scattering of hadrons are well studied in experiment. Both the energy behavior and the dependence on transferred momenta of the differential cross sections are analyzed. Despite many proposed theoretical models, no satisfactory description of the whole sample of experimentally observed features has yet appeared. Most models are purely phenomenological and deal with some sets of adjustable parameters.

The only rigorous requirement imposed on the elastic scattering amplitude follows from the unitarity condition $SS^+ = 1$ mandatory for the S-matrix in any field theory. However, in absence of equations of fundamental theory, one has to use it within definite assumptions and in limited regions of kinematical variables to get any reliable conclusions. In particular, it can be exploited for obtaining some results concerning the behavior of the real and imaginary parts of the elastic scattering amplitude.

In general, the unitarity condition for the elastic scattering amplitude $A(p,\theta)$ is written in a form

$$\mathrm{Im}A(p,\theta) = I_2(p,\theta) + F(p,\theta) = \frac{1}{32\pi^2} \times \iint d\theta_1 d\theta_2 \frac{\sin\theta_1 \sin\theta_2 \mathrm{Im}A(p,\theta_1)\mathrm{Im}A(p,\theta_2)(1+\rho_1\rho_2)}{\sqrt{[\cos\theta - \cos(\theta_1+\theta_2)][\cos(\theta_1-\theta_2) - \cos\theta]}} + F(p,\theta). \quad (1)$$

Here, p and θ denote the momentum and the scattering angle in the center of mass system. ρ_i's take into account the real parts at the corresponding angles. The region of integration over angles in Eq. (1) is given by the conditions

$$|\theta_1 - \theta_2| \le \theta, \quad \theta \le \theta_1 + \theta_2 \le 2\pi - \theta. \quad (2)$$

The integral term represents the two-particle intermediate states of the incoming particles. The function $F(p,\theta)$, called following Ref. [1] as the overlap function, represents the shadowing contribution of the inelastic processes to the elastic scattering amplitude. It determines the main structure in the shape of the diffraction peak and is completely non-perturbative so that only some phenomenological models pretend to describe it. Therefore, the unitarity condition is practically useless at very small angles but can be effectively used outside the diffraction cone as we show below.

The elastic scattering proceeds mostly at small angles. The diffraction peak has a Gaussian shape in the scattering angles or exponentially decreasing as the function of the transferred momentum squared

$$\frac{d\sigma}{dt}\left(\frac{d\sigma}{dt}\right)^{-1}_{t=0} = e^{Bt} \approx e^{-Bp^2\theta^2}. \quad (3)$$

The four-momentum transfer squared is

$$t = -2p^2(1-\cos\theta) \approx -p^2\theta^2 \approx -p_t^2, \quad (4)$$

where p_t is the transverse momentum. At large energies the forward scattering amplitude has a small real part as known both from experiment and from the dispersion relations. Then the elastic scattering in this region labeled by the subscript d can be described by the amplitude

$$A_d(p,\theta) = 4ip^2\sigma_t e^{-Bp^2\theta^2/2}(1-i\rho_d) \quad (5)$$

with a proper optical theorem normalization in the forward direction to the total cross-section σ_t and a small correction due to the real part.

Now, let us consider the integral term I_2 outside the diffraction peak. Because of the sharp fall off of the amplitude (5) with angle, the principal contribution to the integral arises from a narrow region around the line $\theta_1 + \theta_2 \approx \theta$. Therefore one of the amplitudes should be inserted at small angles within the cone while another one is kept at angles outside it. Thus, inserting Eq. (5) for one of the amplitudes in I_2 and integrating over one of the angles the inhomogeneous linear integral equation is obtained:

$$\mathrm{Im}A(p,\theta) = \frac{p\sigma_t}{4\pi\sqrt{2\pi B}} \times \\ \times \int_{-\infty}^{+\infty} d\theta_1 e^{-Bp^2(\theta-\theta_1)^2/2}(1+\rho_d\rho_l)\mathrm{Im}A(p,\theta_1) + \\ + F(p,\theta). \quad (6)$$

It can be solved analytically (for more details see [2, 3]) with the assumptions that the role of the overlap function $F(p,\theta)$ is negligible outside the diffraction cone[1)] and the real parts may be replaced by their average values in the diffraction peak ρ_d and outside it ρ_l, correspondingly. Let us stress once more that the Gaussian shape (5) of the amplitude has been only used at rather small angles in accordance with experimental data.

One gets the analytical solution as the eigenfunction of the homogeneous integral equation with $F(p,\theta)=0$

$$\mathrm{Im}A(p,\theta) = C_0(p)e^{-\sqrt{2B\ln\frac{Z}{1+\rho_d\rho_l}}p\theta} + \\ + \sum_{n=1}^{\infty} C_n(p)e^{-[\mathrm{Re}b_n(p)]p\theta}\cos[|\mathrm{Im}b_n(p)|p\theta - \phi_n], \quad (7)$$

where $Z = 4\pi B/\sigma_t$ and

$$b_n \approx \sqrt{2\pi B|n|}(1 + i\,\mathrm{sign}n), \quad n = \pm 1, \pm 2, ... \quad (8)$$

The solution contains the exponentially decreasing with θ (or $\sqrt{|t|}$) term (Orear regime!) with imposed on it damped oscillations.

Note that the solution predicts the dependence on $p\theta \approx \sqrt{|t|}$ but not the dependence on the collision energy! There are no zeros on the t-axis unless the amplitudes of oscillations $C_n(p)$ become extremely large.

Namely this expression was successfully used to fit the experimental data about the elastic scattering differential cross-section outside the diffraction cone (in the Orear regime region) at comparatively low energies in Ref. [6] and in Ref. [4] at the LHC energy 7 TeV. In the latter case, the value of $Z = 4\pi B/\sigma_t$ is so close to 1 at 7 TeV that the first term is very sensitive to the ratio ρ_l outside the diffraction peak. Thus, it became possible for the first time to estimate ρ_l from fits of experimental data and it happened to be quite large ($\rho_l \approx -2.1$). Concerning the ratio ρ_d it was chosen as prescribed by the dispersion relations for its value at $t = 0$ [7, 8] ($\rho_d = \rho_0 \approx 0.14$).

Comparing the values of ρ_d and ρ_l, one is tempted to understand such a large difference between them. The only guess, we have at present, is obtained from the asymptotical formula derived in Ref. [9] which relates the behaviors of the real and imaginary parts at nonzero transferred momenta t in a following way

$$\rho(t) = \rho_0\left\{1 + \frac{t[d\mathrm{Im}\,A(t)/dt]}{\mathrm{Im}\,A(t)}\right\}. \quad (9)$$

This relation can be explicitly demonstrated now if one uses the first term of the imaginary part of the scattering amplitude at fixed $t < 0$ given by Eq. (7) at finite energies and neglects other terms which decrease much faster with angles. The result is

$$\rho(t) = \rho_0(1 - a\sqrt{|t|}/2), \quad (10)$$

where

$$a = \sqrt{2B\ln\frac{Z}{1+\rho_d\rho_l}}. \quad (11)$$

We note that ρ passes through zero and changes sign at $|t_0| = 4/a^2$. This agrees with the general theorem on the change of sign of the real part of the high-energy scattering amplitude which has been proven first in Ref. [10]. Estimates at 7 TeV [4] show that $|t_0| \approx 0.3\,\mathrm{GeV}^2$.

However, this regime of the unlimited decrease of ρ with $|t|$ does not look satisfactory. And really it can be damped if one does not replace $\rho(t)$ by its average value in the Orear region ρ_l but assumes that its t-dependence may be left intact directly in the solution and differentiates it according to (9) inserting there the first term of (7). Then the following differential equation is obtained

$$\frac{dv}{dx} = -\frac{v}{x} - \frac{2}{x^2}\left(\frac{Ze^{-v^2}-1}{\rho_0^2} - 1\right). \quad (12)$$

Here, $x = \sqrt{2B|t|}$, $v = \sqrt{\ln[Z/(1+\rho_0\rho(t))]}$.

As awaited, the resulting shape of $\rho(t) = (Ze^{-v^2} - 1)/\rho_0$ obtained as the solution of this equation [11] has a single zero at $|t_0| \approx 0.3\,\mathrm{GeV}^2$ and, what is really impressive, it steeply decreases in the Orear region of $0.3 < |t| < 1.4\,\mathrm{GeV}^2$ approaching the large negative saturation value $\rho(|t| \to \infty) = -1/\rho_0 \approx -7.25$ (at 7 TeV) for high transferred momenta $|t|$ (see Fig. 1 in [11]). Note that $f_\rho = 1 + \rho_0\rho(t)$ tends to 0 there.

The bold usage of this procedure for derivation of the equation (12) with $\rho(t)$ inserted directly in the solution is, nevertheless, not satisfactory as well. The two above possibilities should be considered as two extremes for the shapes of $\rho(t)$.

Strictly speaking, the behavior of $\rho(t)$ should be taken into account primarely inside the integral. Then, inserting the expression (9) in place of ρ_l in Eq. (6) and

[1)]The results of the papers [4, 5] give strong support to this assumption.

integrating by parts we derive the following linear integral equation

$$\mathrm{Im}A(x) = \frac{1}{Z\sqrt{\pi}} \times$$
$$\times \int_{-\infty}^{+\infty} dye^{-(x-y)^2}[1 + 0.5\rho_0^2 + \rho_0^2(y-x)y]\mathrm{Im}A(y) \quad (13)$$

with $F(p,\theta) = 0$ and new variables $x = \sqrt{B/2}p\theta$; $y = \sqrt{B/2}p\theta_1$.

The kernel of this equation is not symmetrical. Its solution has not yet been obtained even numerically. However, one can get some preliminary asymptotical estimates from it.

In the preasymptotical energy region we got [3] the Orear regime $\mathrm{Im}A \propto \exp(-ap\theta) \approx \exp(-ap_t)$ with the exponential fall off of the amplitude as a function of angles. Therefore, let us look for the solution of the equation (13) in the form $\mathrm{Im}A(x) = \exp(-ax\sqrt{2/B})\phi(x)$. The Gaussian exponent shifts to $x - y - a/\sqrt{2B}$. Replacing it by the δ-function with this argument, one gets the equation in the finite differences

$$\phi(x) = Z^{-1}e^{a^2/2B}\left[1 + 0.5\rho_0^2\left(1 + \frac{a^2}{B} - ap_t\right)\right] \times$$
$$\times\, \phi\left(x - \frac{a}{\sqrt{2B}}\right). \quad (14)$$

Again, we can not solve it directly but get the important conclusion about the zeros of the imaginary part of the amplitude. The expression in the square brackets is equal to zero at

$$p_{t0} = \frac{2}{a\rho_0^2}[1 + 0.5\rho_0^2(1 + a^2/B)] \approx \frac{2}{a\rho_0^2}. \quad (15)$$

With the present day values of B, a, ρ_0^2 this zero would appear at extremely large $p_{t0} \approx 20$ GeV. However, zeros of the imaginary part of the amplitude in the Orear region just above the diffraction cone might appear as zeros of $\phi(x)$ itself. This result does not contradict to the above statement about absence of zeros in case of small oscillatory terms in the solution of the homogeneous linear integral equation.

Moreover, the equation tells us that $\phi(x)$ and, consequently, the imaginary part of the amplitude may possess zeros at $x_n = x_0 + \frac{a}{\sqrt{2B}}$. On the p_t-axis these zeros would be placed at rather short distances one from another.

In view of smallness of terms proportional to ρ_0^2 in Eq. (13) the effective values of a in the Orear region hardly change very much compared to Eq. (11) with effective ρ_l being rather close to $-\rho_0$ with the factors of the order of 1, i.e. closer to (10) than to (12). Then the black disk limit with Z tending to 0.5 would ask (see Ref. [12]) for the oscillatory behavior of the imaginary part of the amplitude, i.e. to zeros of $\phi(x)$ appearing in the Orear region.

Let us remind that this limit implies an important asymptotical relation between the total cross section and the diffractive slope

$$\sigma_t = 8\pi B. \quad (16)$$

At 7 TeV the coefficient in front of B is still twice smaller. However, if the preasymptotical power-like increase of the total cross section accompanied by a slower rise of the slope persists, the tendency to this limit is promising.

However, it is premature to make the final conclusion until the exact solution of the equation (13) is obtained.

To conclude, the unitarity condition provides important predictions and information about the behavior of the elastic scattering amplitude outside the diffraction cone at present energies and in asymptotics. However, further studies are needed to seek the solution of the equation with the nonsymmetrical kernel first derived above.

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