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Blow-up instability in shallow water flows with horizontally-nonuniform density

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The mechanisms of instability, whose development leads to the occurrence of the collapse (blow up), have been studied in the scope of the rotating shallow water flows with horizontal density gradient. Analysis shows that collapses in such models are initiated by the Rayleigh–Taylor instability and two scenarios are possible. Both the scenarios evolve according to a power law $(t_0 - t)^\gamma$, where t_0 is the collapse time, with $\gamma = -1, -2$, and $\gamma = -2/3, -1$ for the isotropic and anisotropic collapses, respectively. The rigorous criterion of collapse is found on the base of integrals of motion.

1. Introduction. Shallow water models are widely used in the description of large-scale motions in the atmosphere, oceans, rivers, avalanches, etc. The need for shallow water approximation arises in many physical situations when the typical horizontal scale of the motion is much larger than the vertical dimension of the flow.

Besides geophysical fluid dynamics, the classical shallow water models can be useful for studying certain astrophysical phenomena. For example, a shallow water analogue was used to describe both implosive phenomena and the shock instability taking place in the collapsing inner core prior to explosion of a protoneutron star [1]. The shallow water model can also describe the dynamics of the tachocline of a star, as was done in [2] and [3] for the tacholine of the Sun.

For shallow water flows, as well as for other nonlinear systems, the problem of stability is central because development of instability determines the possible final regimes realized in the flows. Various scenarios of instability exist, and one of them is the collapse [4, 5]. This phenomenon implies formation of finite-time singularities and is a rather universal mechanism by which instabilities manifest themselves in nonlinear systems [6–12].

The basic premise of this paper assumes that development of large-scale instability leads to disintegration of flow and to occurrence of particle-like fluid fragments. In next stages, these quasi-regular formations play a role of structural elements from which it is possible to compile an overall picture of the instability up to the final stage when collapse initiating small-scale turbulence is involved in the game.

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The main goal of this work is to study the self-similar scenarios of collapses in rotating shallow water flows with horizontal density gradient. Whether or not the instability leads to collapse depends on the model specificity and initial conditions and it can be established on the basis of integral criteria. For conservative models, such criteria are commonly intimately related to integrals of motion and open a simple way to study power laws under which finite-time singularities form in flows.

2. Shallow water model with horizontally-nonuniform density. We consider a two-dimensional field model whose evolution is described by the equations

$$\partial_t u_i + u_k \partial_k u_i - 2\Omega e_{ik} u_k = \frac{1}{2} h \partial_i \tau - \partial_i (h\tau), \qquad (1)$$

$$\partial_t h + \partial_k (h u_k) = 0, \qquad (2)$$

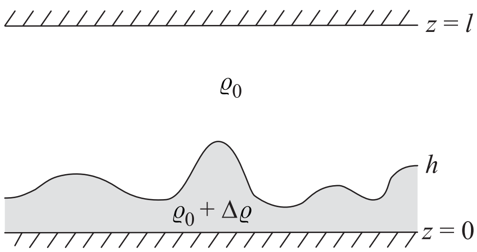
$$\partial_t \tau + u_k \partial_k \tau = 0. \qquad (3)$$

Here, the notations are as follows: $x_i = (x_1, x_2)$ are the Cartesian coordinates; $\partial_t = \partial/\partial t$, $\partial_i = \partial/\partial x_i$; e_{ik} is the unit antisymmetric tensor, $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$; u_i are horizontal components of depth-averaged velocity in layer; h is its thickness. Since Ω is constant angular velocity with which the layer is rotating about the vertical axis, the term $2\Omega e_{ik} u_k$ implies components of Coriolis acceleration. The physical significance of field variable τ depends on the model specification.

One of the simplest specifications [13] suppose that the layer is bounded above by a vacuum (free boundary condition). In this case τ has meaning of the buoyancy and is defined as $\tau = g(1 + \Delta\varrho/\varrho_0)$, where g is gravity and $\Delta\varrho$ is the density deviation from the background value ϱ_0. It must be emphasized that in this specification τ, just as the thickness h, is only a positive function of the horizontal coordinates and time.

Note that in the case when $\Delta\varrho = 0$, so that $\tau = g$ is constant, Eqs. (1)–(3) reduce to the usual shallow water equations describing motion under the influence of gravity g.

The next specification removing the restriction $\tau > 0$ can be justified within the framework of two-layer model. As shown in Figure, this model supposes that



The shallow water model with a relative buoyancy of alternating sign

two incompressible fluids with densities ϱ_0 and $\varrho_0 + \Delta\varrho$ are separated by surface $z = h(x_1, x_2, t)$ and contained between two rigid parallel planes $z = 0$ and $z = l$. If the horizontally-nonuniform density jump $\Delta\varrho$ between the fluids is small and the lower layer is sufficiently thin, so inequalities $\Delta\varrho/\varrho_0 \ll 1$ and $h/l \ll 1$ hold, then the shallow water approximation again leads us to Eqs. (1)–(3). These equations describe depth-averaged flow only in the lower layer, but now the variable $\tau = g\Delta\varrho/\varrho_0$ is the relative buoyancy and therefore may take any sign.

In those cases when density variations are produced only by temperature ones ΔT and are linearly connected, the relative buoyancy can be computed as $\tau = -g\beta\Delta T$, where β is thermal expansion coefficient. This parametrization allows to study heating and cooling effects in shallow water models (see, as an example, [14–16]).

In order to understand how the fluid moves as a whole, it is helpful to consider the evolution of the center of mass. Since integral of total mass

$$Q = \int d\mathbf{x}\, h, \tag{4}$$

is constant, we can find directly from Eqs. (1)–(3) that coordinates of the center of mass X_i and components of the total momentum P_i defined as

$$X_i = Q^{-1} \int d\mathbf{x}\, h x_i, \quad P_i = \int d\mathbf{x}\, h u_i,$$

are governed by the equations

$$\partial_t X_i = Q^{-1} P_i, \quad \partial_t P_i = 2\Omega e_{ik} P_k. \tag{5}$$

Equations (5) can be easily integrated to obtain

$$P_1 = P_0 \sin(2\Omega t), \quad P_2 = P_0 \cos(2\Omega t), \tag{6}$$

$$X_1 = X_{01} - \frac{P_0}{2Q\Omega} \cos(2\Omega t), \tag{7}$$

$$X_2 = X_{02} + \frac{P_0}{2Q\Omega} \sin(2\Omega t), \tag{8}$$

where X_{01}, X_{02} are integration constants, and the modulus of momentum $P_0 = |\mathbf{P}|$ is the constant of motion.

Relations (7) and (8) say that, subject to $\Omega = \mathrm{const} \neq 0$, the center of mass moves around a circle of radius $P_0(2Q\Omega)^{-1}$ with constant angular velocity 2Ω.

3. Integral criteria for collapses. Suppose that the development of instability leads to the formation of a singularity in a point. This means that with the lapse of time, almost the entire fluid is localized in the vicinity of this point. Thus, the collapse point coincides with the center of mass and it is natural to describe collapsing fluid fragments in the center of mass reference frame $\mathbf{x}'$, where the collapse point $\mathbf{x}' = 0$ is immovable.

Since the primed and unprimed coordinates and velocities are connected by the transformation

$$\mathbf{x} = \mathbf{X} + \mathbf{x}', \quad \mathbf{u} = Q^{-1}\mathbf{P} + \mathbf{u}',$$

which leaves invariant Eqs. (1)–(3), in order not to change the notations, it is possible to put from the very beginning that $\mathbf{P} = 0$ and $\mathbf{X} = 0$.

As an indicator of the isotropic collapse, we will use the positive definite integral

$$I = \int d\mathbf{x}\, h\mathbf{x}^2, \tag{9}$$

which has a very simple physical meaning of the moment of inertia with respect to the center of mass. Here and elsewhere we will assume that integrals are taken over the whole area occupied by a collapsing fluid fragment.

On the ground of Eqs. (1)–(3), it is easy to show that I and integrals

$$M = \int d\mathbf{x}\, h\,(x_1 u_2 - x_2 u_1), \quad V = \int d\mathbf{x}\, h\, x_i u_i,$$

which have meaning of the kinetic moment and the virial for the fluid, obey a closed system of equations

$$\partial_t I = 2V, \quad \partial_t V = 2H + 2\Omega M, \quad \partial_t M = -2\Omega V, \tag{10}$$

where the total energy

$$H = \frac{1}{2} \int d\mathbf{x}\, \left(h\mathbf{u}^2 + h^2\tau\right)$$

is a constant of motion.

Equations (10) can be integrated to obtain

$$I = \Omega^{-2}\left(H + \Omega m\right) - \frac{V_0}{\Omega}\cos(2\Omega t), \qquad (11)$$
$$M = -\Omega^{-1}H + V_0\cos(2\Omega t), \quad V = V_0\sin(2\Omega t),$$

where m and V_0 are two more constants of motion

$$m = M + \Omega I, \quad V_0^2 = \left(M + \Omega^{-1}H\right)^2 + V^2. \qquad (12)$$

The isotropic collapse implies a specific temporal evolution when the positive quantity I decreases with increasing t and reaches the value $I = 0$ at a point $t = t_0 > 0$ in the final stage.

Because in the course of time the function J becomes zero only if

$$\left(H + \Omega m\right)^2 \le \Omega^2 V_0^2, \qquad (13)$$

this condition is the criterion for collapse in the rotating shallow water models with a horizontally-nonuniform density.

Representing the total energy H as the sum of kinetic and potential energies

$$H = K + \Pi, \quad K = \frac{1}{2}\int d\mathbf{x}\, h\mathbf{u}^2, \quad \Pi = \frac{1}{2}\int d\mathbf{x}\, h^2\tau,$$

and using (12), we can rewrite inequality (13) in an equivalent form

$$2I\left(\Pi + K'\right) \le V^2. \qquad (14)$$

Here

$$K' = K + \Omega M + \frac{\Omega^2}{2}I^2 = \frac{1}{2}\int d\mathbf{x}\, h\mathbf{u}'^2$$

is the positive definite quantity which can be treated as the kinetic energy in the rotating frame reference.

Since in this frame reference $u'_i = u_i + \Omega e_{ik}x_k$, and hence $x_i u_i = x_i u'_i$, owing to the Cauchy inequality $V^2 \le 2IK'$, we can find from (14) the following condition

$$\Pi \le 0. \qquad (15)$$

Thus, the isotropic collapse occurs if $\Pi < 0$. The only way to provide this condition is by appropriately choosing the initial distribution for field τ which, unlike h, can be signalternating. Negative values of Π indicate directly that the collapse is initiated by the Rayleigh–Taylor instability.

4. Power laws of self-similar collapses. As known [17], self-similar solutions are intermediate asymptotics of non-degenerate problems and are very useful in studying the final stages of strongly nonlinear processes, when the system forgets about details related to the initial data and its behavior depends on the integrals of motion.

For any dynamical system, the existence of self-similar solutions reflects the existence of fundamental internal symmetries and allows us to judge the tendencies in the development of the instability at the final stage. This type of solutions is of particular importance for studying the phenomenon of collapse – the formation of a singularity in a finite time [5, 18].

Isotropic collapses. Suppose that the collapse has isotropic character and demonstrates self-similar behaviour in the vicinity of the point $\mathbf{x} = 0$ so that

$$h(\mathbf{x}, t) = b f(\mathbf{x}'), \quad \mathbf{x}' = \beta^{-1}\mathbf{x},$$

where $f(\mathbf{x}')$ is a shape factor and $b(t)$ and $\beta(t)$ are functions of time.

Then, after integrating in (4) and (9), we can write

$$Q = b\beta^2 \int d\mathbf{x}'\, f(\mathbf{x}'), \quad I = b\beta^4 \int d\mathbf{x}'\, f(\mathbf{x}')\mathbf{x}'^2.$$

Eliminating the function b, we find

$$I = \beta^2 C. \qquad (16)$$

Here, the positive constant C is dependent on the shape factor f only and is defined as

$$C = Q\left[\int d\mathbf{x}'\, f(\mathbf{x}')\right]^{-1} \int d\mathbf{x}'\, f(\mathbf{x}')\mathbf{x}'^2.$$

On the other hand, since in the course of collapsing, the function J asymptotically (as $t \to t_0$) tends to zero, expanding the right part of (11) in powers of $(t_0 - t)$, we approximately obtain

$$H + \Omega m = V_0\Omega\cos(2\Omega t_0),$$
$$I \approx a_1\left(t_0 - t\right) + a_2\left(t_0 - t\right)^2 + \ldots, \qquad (17)$$

where

$$a_1 = 2\sqrt{V_0^2 - \Omega^{-2}\left(H + \Omega m\right)^2}, \quad a_2 = 2\left(H + \Omega m\right).$$

Thus, the comparison of (16) with (17) allows us to make the following conclusions.

1. If $a_1 \ne 0$, i.e., the inequality (13) is strict, then the isotropic collapse obeys laws

$$\beta \sim \left(t_0 - t\right)^{1/2}, \quad h \sim \beta^{-2} \sim \left(t_0 - t\right)^{-1}. \qquad (18)$$

2. But if $a_1 = 0$, i.e., the inequality (13) turns into equality, then, instead of (18), we obtain the laws

$$\beta \sim t_0 - t, \quad h \sim \beta^{-2} \sim \left(t_0 - t\right)^{-2}. \qquad (19)$$

Anisotropic collapses. When collapse loses the radial symmetry, power laws (18), (19) become unusable. In such situations, a collapsing liquid fragment contracts into a line segment rather than into a point. For this reason, using positive definite integral (9) to test the anisotropic collapse is doomed to failure.

According to [12], anisotropic collapses in $2D$-model have the same exponents as one-dimensional ones in flat model. The only difference is that in the flat model a liquid fragment shrinks not into a line segment but into an infinite axis perpendicular to the flow plane. Thus collapses in the flat model represent an idealization.

In the one-dimensional case, instead of Eqs. (10), we have

$$\partial_t I = 2V, \quad \partial_t V = H + K, \tag{20}$$

where

$$I = \int dx\, hx^2, \quad V = \int dx\, hxu, \quad H = \Pi + K = \text{const},$$
$$\Pi = \frac{1}{2}\int dx\, h^2\tau, \quad K = \frac{1}{2}\int dx\, hu^2$$

are integrals in one dimension.

Thus, we have two Eqs. (20) in the three variables I, V, and K. In order to close the system, we need one more equation. It can be obtained from the Cauchy inequality

$$V^2 \leq 2IK$$

under the assumption that, for the collapsing solutions, this inequality asymptotically (as $t \to t_0$) turns into an equality.

As a result, after using (16) which remains valid for the one-dimensional case too, we arrive at the single equation

$$\beta\partial_t^2\beta + \frac{1}{2}\left(\partial_t\beta\right)^2 - \frac{H}{C} = 0. \tag{21}$$

Equation (21) has two power-law solutions. One of them,

$$\beta \sim (t_0 - t)^{2/3}, \quad h \sim \beta^{-1} \sim (t_0 - t)^{-2/3},$$

makes sense if $H = 0$, while the other,

$$\beta \sim (t_0 - t)\,, \quad h \sim \beta^{-1} \sim (t_0 - t)^{-1}\,,$$

is realized if $H > 0$.

Note that due to conservation of the integral $m = M + \Omega I = \text{const}$, the line segment which accumulates fluid in the case of anisotropic collapse should rotate with a constant angular velocity.

5. Conclusions. We now summarize the main results of the work. The basic goal of this paper was to study power-law collapses in $2D$-shallow water flows with horizontal density gradients. We discuss the model specifications and formulate the governing equations.

According to the rigorous integral criterion, the isotropic collapse is initiated only if the distribution of density or of temperature is such that the potential energy integral is non-positive. Therefore, the mechanism that invokes the collapse is the Rayleigh–Taylor instability.

In our opinion [7–12], this phenomenon arises at the final stage when the development of instability has led to disintegration of the strongly perturbed flows. Once localized (drop-like) fragments are formed, the collapse eventually occurs and leads to the formation of finite-time singularities.

We also discuss the self-similar scenarios of collapses. Analysis shows that two collapse scenarios are possible depending on whether the drop-like fragment is contractible into a segment or into a point. If the collapse is isotropic, it leads to point space singularities in which the height h behaves as $h \sim (t_0 - t)^{-1}$ or $h \sim (t_0 - t)^{-2}$. In the anisotropic case, singularities are located at line segments and the collapse can follow the slower law $h \sim (t_0 - t)^{-2/3}$.

Since the singularities produce power-law tails in the short-wave range of spectrum, the study of collapses provides the key to understanding of strong turbulence.

Note that the formation of finite-time singularities is an idealization. For this reason, the collapse mechanism considered here must be treated as an initial stage of instability for the real physical system. At the next stage, when small scales come into the play, the collapse dynamics goes beyond the validity range of the shallow water approximation and we should take into account the influence of dispersion effects. In this case the stabilization of collapses and the formation of soliton-like structures is a possible scenario, but not a necessary one. In particular, self-similar collapses initiated by a nonlinear dispersion are considered in works [9–12].

In the present brief work we do not touch the problem of finding structural elements which correspond to the self-similar collapses. This problem will be considered by us in a subsequent paper.

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