Whispering gallery like modes along pinned vortices

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We study sound propagation in stationary and locally irrotational vortex flows where the circulation is wound around a long (rotating) cylinder, using Unruh's formalism of acoustic space-times. Apart from the usual scattering solutions, we find anomalous modes which are bound to the vicinity of the cylinder and propagate along its axis – similar to whispering gallery modes. These modes exist for subsonic and supersonic flow velocities. In the supersonic case (corresponding to an effective ergo-region in the acoustic space-time), they can even have zero frequency $\omega = 0$ and thus the associated quasi-particles with $E = \hbar\omega = 0$ are easy to excite from an energetic point of view. Hence they should be relevant for the question of stability or instability of this setup.

Introduction. The full characterisation of sound modes propagating within a given flow profile is a major problem in fluid dynamics and often reveals very rich physics. Even for stationary flows, which admit a separation ansatz where the linear perturbations can be labelled by their conserved frequency ω, this problem is highly non-trivial: For *static* systems, depending on what sort of scenario is considered, the dynamics of perturbations is governed by equations of Schrödinger $i\partial_t\psi = \mathcal{H}\psi$ or d'Alembert $\partial_t^2\Phi = \mathcal{D}\Phi$ type. In such cases the full characterisation of solutions follows from the spectral analysis of the differential operators $\mathcal{H}$ or $\mathcal{D}$. For *stationary* systems, however, the following inherent difficulty appears. The equations assume the form $\partial_t^2\Phi + \mathcal{A}\,\partial_t\Phi = \mathcal{B}\Phi$, where the two operators $\mathcal{A}$ and $\mathcal{B}$ do not commute in general, and therefore their spectral content has no direct significance for the problem at hand. As a result, questions like the completeness of solutions or the existence of unstable modes with $\Im(\omega) < 0$ are far more difficult to address.

The wave equation for sound in a locally irrotational and stationary background flow has the form mentioned above, $\partial_t^2\Phi + \mathcal{A}\,\partial_t\Phi = \mathcal{B}\Phi$. Precisely the same structure arises for scalar fields propagating in a stationary space-time. Moreover, as discovered by Unruh [1], there is an exact correspondence between the two cases. Let us consider a fluid with density ϱ and velocity $\mathbf{v}$, whose pressure p is a function of ϱ only: $p = p(\varrho)$, i.e., the fluid is barotropic. The perturbations, i.e., sound waves, can be parametrised by a single potential Φ via $\delta\mathbf{v} = \boldsymbol{\nabla}\Phi$ and $\delta\varrho = \varrho\dot{\Phi}/c_s^2$. Neglecting viscosity, they obey the same wave equation as a scalar field in a curved space-time described by the effective acoustic metric [1]

$$ds^2 = \frac{\varrho}{c_s}\left[c_s^2 dT^2 - (d\mathbf{R} - \mathbf{v}\,dT)^2\right], \qquad (1)$$

where T, $\mathbf{R}$ are the laboratory coordinates and c_s is the speed of sound: $c_s^2 = dp/d\varrho$. This mathematical correspondence allows us make use of standard geometrical tools and concepts, usually employed in general relativity, in fluid dynamics.

Vortex flow. In the following, we consider a stationary and locally irrotational $\boldsymbol{\nabla}\times\mathbf{v} = 0$ flow around a long cylinder, cf. Fig. 1. Aligning the coordinate Z-axis

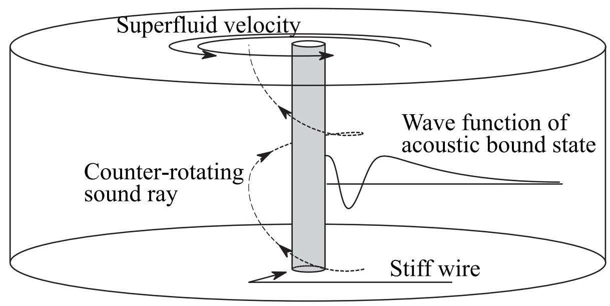

Fig. 1. Sketch (not to scale) of the considered setup

with the symmetry axis of the cylinder, we assume the flow velocity to be $\mathbf{v} = v(R)\mathbf{e}_\varphi$ in cylindrical coordinates Z, R, φ. The condition $\boldsymbol{\nabla}\times\mathbf{v} = 0$ then implies $\mathbf{v} = \mathbf{e}_\varphi\kappa/R$, where κ determines the circulation.

For normal fluids, such a profile approximates the stationary flow around a rotating cylinder: If the fluid is incompressible $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$, the above velocity profile provides an exact solution of the Navier–Stokes equations (similar to a tornado away from the core). While this solution gets modified in more realistic models of normal fluids, for super-fluids (such as ^{4}He II), vorticity can only occur in the form of vortices with a quantized circulation, which is thus topologically stabilised and not so easy to create (or destroy). The flow profile $\mathbf{v} = \mathbf{e}_\varphi\kappa/R$ then corresponds to a vortex which is pinned around a long cylinder (e.g., wire), where κ is an integer multiple of the circulation quantum $\kappa_0 = \hbar/M_{\text{He}}$ [2].

In fluid dynamics, it is often useful to express the problem in terms of dimensionless quantities (such as Rossby number) in order to exploit the scaling symmetries. Here, we do the same and choose the sound velocity at infinity $c_s(R \uparrow \infty) \equiv c_\infty$ as reference scale. Using the circulation κ we introduce a length unit $\mathfrak{L}_1 = \kappa/c_\infty$ and an angular frequency unit $\omega_1 = c_\infty^2/\kappa$. By rescaling the laboratory coordinates T and $\mathbf{R}$, to get dimensionless t and $\mathbf{r}$, we also find that (1) becomes

$$ds^2 = \\ = \frac{\rho}{c}\left[\left(c^2 - \frac{1}{r^2}\right) dt^2 + 2dt\, d\varphi - dr^2 - r^2 d\varphi^2 - dz^2\right], \quad (2)$$

where $c(r) = c_s(r)/c_\infty$ is the dimensionless sound speed which approaches unity at infinity. Similarly, the normalised density reads $\rho(r) = \varrho(r)/\varrho_\infty$. For typical fluids, both of the above functions decrease when approaching the cylinder where the velocity $\mathbf{v}$ increases and thus the pressure p drops. In case of ^{4}He II this decrease is below 20% for typical parameters.

The coordinates t, z, and φ have their standard ranges, but r is restricted to $r \in (r_w, +\infty)$, where r_w is the rescaled wire radius. Note that the acoustic space-time (2) may possess an *ergo-region* [3], where $g_{00} < 0$. This is the case for small enough radii $r < 1/c$, if such r are in the allowed range, that is, if $r_w < 1/c(r_w)$. From the laboratory point of view appearance of ergo-region means that the flow velocity $\mathbf{v}$ exceeds the local speed of sound c_s in a region. As we shall see, this has profound consequences for the sound modes.

Geometric acoustics. Sound propagation in the presence of an irrotational background flow can conveniently be described using Unruh's formalism [1]. Sound modes in the vortex profile are solutions of the wave equation of a massless scalar field in the space-time with the metric (2). However, before investigating the full wave equation, let us get some insight via the WKB approximation – which amounts to studying sound rays. They are null geodesics in the space-time (2), and as we shall see they are easily found. We search for curves $x^a(\tau)$ satisfying the geodesic equation $\ddot{x}^a + \Gamma^a_{bc}\dot{x}^b\dot{x}^c = 0$ with $\dot{x}^b = dx^b/d\tau$ and $\ddot{x}^a = d^2x^a/d\tau^2$, where the affine parameter τ can be chosen arbitrarily. We find that the problem is reduced to quadratures due to existence of four independent first integrals. The space-time (2) admits three symmetries with the Killing vectors ∂_t, ∂_φ, and ∂_z. Via the Noether theorem, this implies the conservation of the energy $E = \Omega(f\dot{t} + \dot{\varphi})$, the angular momentum $J = \Omega(r^2\dot{\varphi} - \dot{t})$, and the axial momentum $P = \Omega\dot{z}$. Here, the notation $\Omega = \rho(r)/c(r)$ and $f = c^2 - 1/r^2$ is introduced for brevity. Together with the null ray condition $\dot{x}_a\dot{x}^a = 0$, we can express all velocities in terms of these first integrals, e.g., $\dot{t} = (E - J/r^2)/(\Omega c^2)$ and $\dot{\varphi} = (E + fJ)/(\Omega c^2 r^2)$ as well as $\dot{z} = P/\Omega$. The remaining radial equation reads

$$\Omega^2\dot{r}^2 + P^2 = \\ = \frac{E^2}{c^2} - \frac{J^2c^2 + 2EJ}{c^2r^2} + \frac{J^2}{c^2r^4} = E^2 - V_{\text{eff}}(r), \quad (3)$$

where we have introduced the effective potential $V_{\text{eff}}(r)$ which also contains the term $E^2(1 - 1/c^2)$.

The sound rays can now be classified by the following arguments. For $r \uparrow \infty$, the effective potential $V_{\text{eff}}(r)$ vanishes. Hence all scattering solutions must have $E^2 \geq P^2$. For $r \downarrow 0$, of the other hand, the effective potential $V_{\text{eff}}(r)$ diverges $V_{\text{eff}}(r \downarrow 0) \downarrow -\infty$ for $J \neq 0$. Thus rays are strongly attracted by the vortex in its vicinity. There is a cut-off in r, however, provided by r_w (wire radius), where the sound rays are reflected. If, due to r_w, the potential $V_{\text{eff}}(r)$ is monotonically *decreasing* for all $r > r_w$, then only scattering solutions ($E^2 \geq P^2$) exist. If, however, there are local minima of $V_{\text{eff}}(r)$ at finite $r \in [r_w, \infty)$, bound rays sitting at those minima (or oscillating around them) will exist. As one may easily infer from the structure of Eq. (3), this can always be achieved by tuning the angular momentum J. Choosing, e.g., $J = -E$, we see that $V_{\text{eff}}(r)$ is strictly negative (assuming $c \leq 1$ everywhere; see below), and that there will exist rays bouncing off r_w indefinitely. Note that these are counter-rotating rays, i.e., propagating against the vortex flow.

In the special case of constant c and ρ (i.e., $\Omega = 1$), these qualitative arguments can be made precise. The maximum of $V_{\text{eff}}(r)$ is at $r_* = \sqrt{2J/(J + 2E)}$ where $V'_{\text{eff}}(r_*) = 0$. Let J, E be fixed and r_w, P adjustable. For $r_w > r_*$ we only have scattering rays, otherwise there exist also bound rays. Generally, for arbitrarily large wire radii r_w, one can find values of J and E for which there exists no maximum of V_{eff} (r_* is imaginary), and therefore bound rays exist. For small J, the radius r_* goes to zero – i.e., bound states require a minimum angular momentum. Finally, for large J, the radius $r_* \to \sqrt{2}$ which is outside the ergo-region at $r = 1$.

Surprisingly, in the special case of constant c and ρ (i.e., $\Omega = 1$) as above, the problem of finding null orbits, $r(\varphi)$, is exactly soluble in terms of elliptic functions (see [4]), the reason being not-more-than quartic dependence of $V_{\text{eff}}(r)$ on $1/r$, as is also the case in the general relativistic Kepler problem, for example. We note that the J^2/r^4-term is crucial for the universal existence of bound states of sound as discussed above. As we shall see later, this remains correct for the full wave equation. However, available treatments [5–7] of the subject of propagation of waves in vortex backgrounds introduce assumptions, which effectively eliminate this term. While this leads to the radial equation of simple Bessel form, it also is the reason why the family of bound states reported upon here is not to be found in the literature [5–7].

Wave acoustics. In what follows, the full wave equation will be considered. To this end, let us first discuss the field expressions for E, J, and P. They can be obtained from the pseudo energy-momentum tensor [8]

$$T_{ab}[\Phi] = (\partial_a \Phi)(\partial_b \Phi) - \frac{1}{2}\, g_{ab}\, g^{cd}\, (\partial_c \Phi)(\partial_d \Phi), \tag{4}$$

where Φ is the velocity potential $\delta \mathbf{v} = \boldsymbol{\nabla}\Phi$ of the sound waves $\delta\mathbf{v}$ and g_{ab} is the acoustic metric (2). For each Killing vector field ξ, we get a conserved field quantity $\Xi = \int dS^a\, T_{ab}\, \xi^b$. This means, in particular, that Ξ is independent of the (laboratory) time t. As usual, invariance under time-translations leads to the conserved energy,

$$E[\Phi] = \int d^3 r\, \frac{\rho}{2c^2} \left(\dot{\Phi}^2 + c^2 [\boldsymbol{\nabla}\Phi]^2 - \frac{1}{r^4} [\partial_\varphi \Phi]^2 \right), \tag{5}$$

where Minkowski scalar product is implied in the term $[\boldsymbol{\nabla}\Phi]^2$, i.e., $[\boldsymbol{\nabla}\Phi]^2 = (\partial_r \Phi)^2 + (\partial_z \Phi)^2 + (\partial_\varphi \Phi)^2/r^2$. The energy functional is positive definite as long as $c_s^2 > \mathbf{v}^2$ everywhere, i.e., $c^2 > 1/r_w^2$. This is not anymore the case if the space-time contains an ergo-region. The two remaining conserved functionals following from the Killing vector fields ∂_z and ∂_φ, are the axial momentum

$$P[\Phi] = \int d^3 r\, \frac{\rho}{2c^2} \left(\partial_t \Phi + \frac{1}{r^2} \partial_\varphi \Phi \right) \partial_z \Phi, \tag{6}$$

and similarly the angular momentum $J[\Phi]$ after replacing $\partial_z \Phi \to \partial_\varphi \Phi$. Another useful concept can be introduced if we admit complex solutions (potentials) Φ. True velocity potentials are always real, of course, and can be derived from the complex Φ as usual via the real (or imaginary) part. Due to the $U(1)$ gauge-invariance of the wave equation for such complex potentials, the Klein–Fock–Gordon *inner product* of two solutions

$$(\Phi_1 | \Phi_2) = \frac{i}{2} \int d^3 r\, \Phi_1^* \overleftrightarrow{\left(\partial_t + \frac{1}{r^2} \partial_\varphi \right)} \Phi_2, \tag{7}$$

with $\Phi_1^* \overleftrightarrow{\partial_a} \Phi_2 = \Phi_1^* \partial_a \Phi_2 - \Phi_2 \partial_a \Phi_1^*$ is conserved, i.e., independent of the Cauchy surface over which it is taken (in particular: independent of t).

Separation ansatz. In view of the symmetries, we consider the following separation ansatz

$$\Phi(t, r, \varphi, z) = \phi(r) \exp\{-i\omega t + i m \varphi + i p_z z\}. \tag{8}$$

For such solutions Φ, the conserved quantities are related to the inner product via $E[\Phi] = \omega(\Phi|\Phi)$, $P[\Phi] = p_z(\Phi|\Phi)$, and $J[\Phi] = m(\Phi|\Phi)$. Since $(\Phi|\Phi)$ and $E[\Phi]$ are always real, solutions with complex frequencies – if they exist – would need to have $(\Phi|\Phi) = 0$ and $E[\Phi] = 0$. In the absence of an ergoregion, $E[\Phi]$ is positive definite and thus all frequencies are real, i.e., the flow is linearly stable [9]. Furthermore, the pseudo-norm $(\Phi|\Phi)$ of our modes (not guaranteed to be positive) and their frequency ω have the same sign in this case. As a result, creation and annihilation operators are associated to modes with positive and negative frequencies, respectively. In the case with an ergo-region, the energy can become negative and hence this property is no longer true. This can lead to interesting and related phenomena such as super-radiance [10] and the Klein paradox [11]. Since a given frequency $\omega > 0$ can be associated to both, creation and annihilation operators, one can have a mixing of the two and thus phenomena like particle creation.

By inserting the above separation ansatz (8) into the wave equation we reduce it to a single ordinary differential equation (in radial direction):

$$\begin{aligned} &\left[-\frac{1}{r\rho} \frac{d}{dr} r\rho \frac{d}{dr} + \omega^2 \left(1 - \frac{1}{c^2} \right) + \right. \\ &\left. + \frac{m^2 c^2 + 2m\omega}{c^2 r^2} - \frac{m^2}{c^2 r^4} \right] \phi = \\ &= \mathcal{H}\phi = \left(\omega^2 - p_z^2 \right) \phi = \lambda \phi. \end{aligned} \tag{9}$$

This main equation of our sound-propagation problem has several interesting features [4]. First of all, because $\rho \to 1$ and $c \to 1$ at $r \uparrow \infty$, the solutions $\phi(r)$ at large r are either oscillating, for $\omega^2 > p_z^2$, or exponentially decaying, for $\omega^2 < p_z^2$. In complete analogy to the ray problem, we call the first type of solutions the *scattering modes*, and the second the *bound-state modes*. Finding these modes is then reduced to an eigenvalue problem $\mathcal{H}\phi = \lambda\phi$ with $\mathcal{H} = \mathcal{D} + \mathcal{V}$, where $\mathcal{D}$ stands for the "kinetic part" involving r-derivatives and $\mathcal{V}$ is the effective potential. Note that $V_{\text{eff}}(r)$ in the sound-ray problem corresponds to $\mathcal{V}$ on identification of E and J with ω and m, respectively. For the scalar product

$$\{\phi_1 | \phi_2\} = \int_{r_w}^{\infty} r dr\, \rho(r)\, \phi_1^*(r) \phi_2(r), \tag{10}$$

the operator $\mathcal{H}$ is self-adjoint if the Neumann boundary condition at r_w is assumed $\phi'(r_w) = 0$, which would just reflect the fact that perturbations can not penetrate the wire. Even though neither is $\mathcal{H}$ the Hamiltonian of the problem, nor is the scalar product (10) distinguished by the geometry, both of these elements suffice for the following general statements. First of all, $\mathcal{H}$ has bound states, $\lambda < 0$, only if one can find test functions ψ such that $\{\psi|\mathcal{H}|\psi\} < 0$. The kinetic part, $\mathcal{D}$, is always non-negative, and therefore bound states can only exist if $\mathcal{V}$ is sufficiently negative. A lower bound on the eigenvalue λ can be obtained via the minimum of $\mathcal{V}(r)$. Then, for general profiles of $c(r)$ and $\rho(r)$ with the aforementioned asymptotics, we can prove the following statements, which are the main result of this paper:

I. Bound states with $\omega = 0$ can only exist if the fluid velocity (as seen by laboratory observers) exceeds the local speed of sound somewhere $c(r) < 1/r$, i.e., if the acoustic space-time has an ergo-region. Otherwise $\mathcal{V}(\omega = 0)$ and thus also $\mathcal{H}$ are non-negative (no bound states).

II. Bound states with $m = 0$ can only exist if c_s becomes sufficiently smaller than c_∞ near the wire. The mechanism for bound states in this case is just the total reflection from the region with larger speed of sound – which can also occur in a non-rotating fluid.

III. Independently of the mechanism of II, caused by the ω^2 term in $\mathcal{V}$, the other terms in $\mathcal{V}$ can only allow for bound states with $m\omega > 0$ (i.e., co-rotating) if the fluid flow is locally supersonic, as in I.

IV. For any radius of the central wire, r_w, there are always bound states for some (possibly large) values of angular quantum number m and frequency ω.

We begin justification of the statements by arguing that quite generally the velocity of sound drops towards the axis of rotation: the Bernoulli theorem ($\mathbf{v}^2/2 + h(\varrho) = \text{const}$ for a free stationary flow) implies that the specific enthalpy $h(\varrho)$ drops towards the axis, because $\mathbf{v}$ increases. Thus the pressure p and the density ϱ must decrease near the wire. For a large family of fluids the Grüneisen parameter, $\propto dc_s/d\varrho$, is positive (e.g., for ^{4}He II at $T = 0$, we have $dc_s/d\varrho \in \in [2.2, 2.9] \times c_s/\varrho$ [12]), and therefore the speed of sound is also a monotonically decreasing function of r. Thus the term $\omega^2(1 - 1/c^2)$ in $\mathcal{V}$ is negative. For sufficiently large ω, one could get bound states for $m = 0$ via total reflection, see point II. However, if the variation of $c(r)$ is small the frequencies required for bound states might be too large for the underlying fluid dynamic description to be valid.

In contrast, the bound states for $m \neq 0$ mentioned in point IV can occur for smaller values of ω, for which fluid dynamics is valid, and might even dominate the macroscopic behaviour of the fluid (as in the instability of the Taylor–Couette flow [13]). Let us prove the existence of states of type IV. To this end, we exploit the freedom of varying parameters in the operator $\mathcal{H}$, while keeping a single test function $\psi(r)$ fixed. In the potential $\mathcal{V}$ we are free to adjust m and ω. The kinetic part of the expectation value of $\mathcal{H}$ is positive, $\{\psi|\mathcal{D}|\psi\} > 0$, and independent of m and ω. On the other hand, for counter-rotating modes, $m\omega < 0$, the expectation value of $\mathcal{V}$ can be made arbitrarily negative: Taking, e.g., $\omega = -m$, we see that $\{\psi|\mathcal{V}|\psi\}$ scales as m^2 and is negative $\{\psi|\mathcal{V}|\psi\} < 0$ for any $\psi \neq 0$. Thus, if m^2 is large enough, we get $\{\psi|\mathcal{H}|\psi\} < 0$, i.e., bound states must exist.

Case of constant ρ and c. The arguments stated above prove the existence of the bound states for large enough m, but they do not provide information about how large m must be for a given setup and how ω depends on p_z, for example. To get this information, we numerically solve Eq. (9) for the special case of constant ρ and c. Since the family of bound states depends on four parameters, (r_w, ω, p_z, m), it is necessary to investigate selected sections of this four-dimensional space. We first present examples illustrating general statements shown above, and subsequently give a more systematic discussion of the physically most-relevant case where the superfluid flow is neither supersonic, nor exceeding the Landau critical velocity anywhere.

Illustrative examples. In Fig. 2, we have plotted an example for the dispersion relations $\omega(p_z)$ of the bound

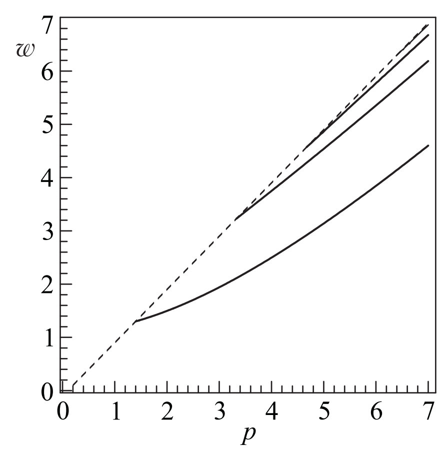

Fig. 2. Typical dispersion relations for counter-rotating bound states; here for the case without ergo-region $r_w = = 1.01$ and $m = -6$. The frequency gap $\omega_{\min} \approx 1.3$ is a generic feature for subsonic flows (no ergo-region)

states in case of a setup just avoiding supersonic flow, i.e. barely avoiding having an ergo-region, with $r_w = 1.01$. (For such a value of r_w general features of dispersion relations are well discernible.) In agreement with statements II and III, we find that only counter-rotating bound states exist, for all values of $m < 0$ (without loss of generality we consider $\omega > 0$). In agreement with statement I, there is a gap for all modes, i.e., a minimum frequency, of $\omega_{\rm min} \approx 1.3$ below which no *bound* states exist. (Note, that the value of $\omega_{\rm min}$ depends on m and r_w, but its existence is generic.) Above this gap, the dispersion relation quickly becomes approximately linear, which means that these modes propagate with the group velocity almost equal c along the vortex. For comparison, we plotted an example with an ergo-region $r_w = 0.3$ in Fig. 3 where the overall change of character of the

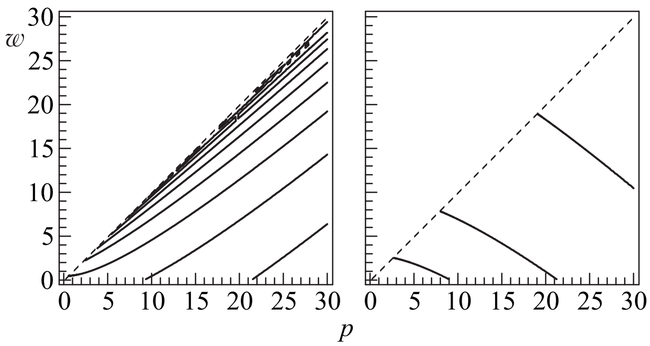

Fig. 3. Typical dispersion relations of bound states for a setup including supersonic flow (with an ergo-region); here for $r_w = 0.3$, left plot: counter-rotating modes with $m = -5$, right plot: co-rotating modes with $m = 5$

spectrum of sound modes is evident. Consistent with point III, families of co-rotating modes $m > 0$ do exist in this case. Furthermore, there are co- and counter-rotating bound states which reach $\omega = 0$, cf. point I. Note that the combined $t \to -t$ and $\varphi \to -\varphi$ symmetry of our set-up implies existence of states with $(-\omega, -m)$ when states with (ω, m) exist, and therefore states with $\omega = 0$ are degenerate, and correspond to both signs of the angular wavenumber m. Let us also remark that in all cases the region with $|\omega| > |p_z|$ is filled with a continuum of scattering states, which are well known. Quantitative properties of the scattering states lead to the precise understanding of the fundamental phenomenon of the Iordanskii force [6, 14, 15] (force acting on a vortex moving against a uniform superflow).

The apparent softening of the spectrum of sound modes in the presence of supersonic flows (ergo-regions), as well as the richness of the family of sound modes is remarkable. This brings about the question of observability of the bound-state modes, and in turn the question of stability of superfluid setups necessary for their existence. The problem of stability can, again, be addressed on different levels of sophistication, including ones respecting the structure of excitations of the superfluid (true quantum many-body excitations, Landau criterion). Within the universality class of fluid dynamics we have shown in [4] that in the absence of ergo-regions the system is at most marginally unstable. In the presence of ergo-regions, however, we conjecture, that the situation is the same (no complex frequencies). This is supported by results of semi-numerical studies in [4]. The dynamics of the (possible) marginal instability needs to be considered on the model-dependent basis; for the superfluid ^{4}He we refer the reader to the extensive discussion in parts VI and VII of [15]. Suffice us to note, that should the superflow velocity be everywhere lower than the Landau critical velocity, current state of understanding of the microscopic theory does not foresee any mechanism for instability.

Background flow below Landau velocity. We present here a discussion of the bound states in setups in which the superflow velocity never exceeds Landau's velocity $v_{\rm L}$. As $v_{\rm L}$ is typically lower than the speed of sound, c, by a factor of ≈ 4 (for Helium), a setup with (dimensionless) $r_w \geqslant 4$ will not exhibit superflows with $v > v_{\rm L}$ (recall that $r_w = 1$ would correspond to a radius where $v = c$). As we shall show, the most serious challenge related to the experimental detection of the bound states comes from their relatively high frequencies. Below we attempt to find conditions where the experimental requirements are the least challenging.

The most efficient way for searching through the 4-dimensional parameter space (r_w, ω, p_z, m) for bound states with lowest ω (in the absence of supersonic flow) is provided by scaling transformations [4]. These stem from the fact that the radial equation for constant $(\rho,\ c)$ can be written in terms of three variables, (x, K, M), only:

$$x^2 = |k/m|\, r^2, \quad K^2 = |mk|, \quad M^2 = m(m + 2\omega), \tag{11}$$

where $k = \sqrt{p_z^2 - \omega^2}$, and the radial equation assumes the very elegant form

$$-\frac{d^2\phi}{dx^2} - \frac{1}{x}\frac{d\phi}{dx} + \frac{M^2}{x^2}\phi + K^2\left[1 - \frac{1}{x^4}\right]\phi = 0, \tag{12}$$

with a Neumann boundary condition $\frac{d\phi}{dx}|_{x_w} = 0$. (This equation is equivalent to the modified Mathieu equation.) Positions of bound states in the space (x_w, K, M) can be interpreted as correspodning to problems with essentially any r_w, but the with remaining parameters (ω, p_z, m) dependent upon the choice (of r_w). For the task at hand we choose $r_w = 4$, and attempt to minimize ω over all allowed bound states. Without restricting generality we set $m > 0$ and find

$$m = \frac{r_w K}{x_w}, \qquad \omega = \frac{1}{2}\left(\frac{M^2}{m} - m\right), \tag{13}$$

so that for every position of the bound state in the (x_w, K, M) space (to be determined), the m and subsequently ω can be computed.

Numerical investigation shows that lowest ω's result from the region of small (x_w, K, M) (each of these parameters $\leqslant 0.5$). In this region, for fixed M, x_w of a bound state depends almost linearly on K and therefore the computed m and ω depend only on the slope of the line, see Fig. 4. The problem of finding the lowest-

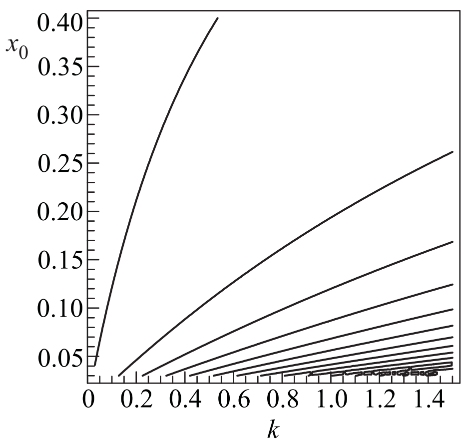

Fig. 4. Positions of bound states (dark lines) in (K, x_w) plane for $M = 0.2$. Linear relation for $x_w(K)$ is evident

frequency bound state is thus well-posed; an interesting solution (not optimal) is found in the $M = 0.2$ cut, with $x_w = 0.15$ and $K = 0.129$ leading to $m = 3.44$. As m must be an integer, we take $m = 4$ corresponding to the choice of $r_w = 4.65 > 4$ (background velocity below the Landau velocity by a margin of 15%). This gives $\omega \approx -2$ (by symmetry there exists a bound state with $m = -4$ and $\omega = 2$), and the "radial wavenumber" is $k = 4 \cdot 10^{-3}$ (i.e. the radial extension of the perturbation is about 10^3 times larger than the wire-radius).

We have therefore found a bound state with $(r_w, \omega, p_z, m) = (4.65, 2, 2, -4)$. All these values are given in *dimensionless* units; corresponding *physical values* depend on the units of length/angular frequency used, which in turn depend on the circulation κ of the background flow. If a number N of circulation quanta κ_0 is wound on the wire, than the corresponding units scale according to

$$\omega_N = \omega_1/N,$$
$$\mathfrak{L}_N = \mathfrak{L}_1 \cdot N. \quad (14)$$

(Recall: $\mathfrak{L}_1 = \kappa_0/c_\infty$, $\omega_1 = c^2/\kappa_0$.) The challenge associated with large frequencies in case of $N = 1$ is the following: the dimensionless frequency ω would need to be multiplied with the unit $\omega_1 = 3.6 \cdot 10^{12}\,\mathrm{s}^{-1}$, which exceeds the roton frequency in ^{4}He, $\omega_{\mathrm{rot}} = 1.15 \cdot 10^{12}\,\mathrm{s}^{-1}$. Scaling with N allows for lowering of this frequency; already for $N = 13$ the physical frequency of the bound state is less than a half of the roton frequency. The linearity of the (bulk) dispersion relation of helium II for such frequencies, and the required wire radius of ($r_w \approx 3\,\mathrm{nm}$), make reliability of the hydrodynamic description plausible, and in turn validate our considerations.

Conclusions. For a pinned vortex where the flow is wound around a cylinder (Fig. 1), we studied sound propagation via Unruh's formalism of acoustic space-times. The formalism has proved to be especially useful in this case, as the crux of the problem is quickly reached and can be studied in depth using well-established methods of classical field theory in curved space-times. On general grounds, we predict the existence of bound states of sound – whispering gallery like modes – based on the geometric acoustics approximation as well as the full wave equation. Assuming the fluid dynamical description to be valid, these bound states should exist for arbitrary non-zero circulations κ (even in the slowly-rotating regime), but their effects are most relevant for relatively compact wires with strong circulation.

As an example of an (in principle observable) bound state we have considered a setup with 13 circulation quanta wound on a wire of radius 3 nm (as in Fig. 1). In this setup the lowest angular frequency of the ($m = -4$, counter-rotating) bound state is $\omega = 5.5 \cdot 10^{11}\,\mathrm{s}^{-1}$, and is lower than half of the roton frequency. The background superflow velocity is everywhere below the Landau critical velocity of 60 m/s (stemming from the roton dip), and the setup does not allow for acoustic ergo-region. We see no reason for expecting standard instability paths to set in this case. Existence of bound states of sound, which are localized on distances of the order of μm from the wire, should lead to enhanced transmission of sound (with frequencies above the gap $\omega_{\min}$) along the wire. In this way the vortex acts as a micro-waveguide, with several branches of the dispersion relation $\omega(p_z, m)$ as in Fig. 2, which are computable (numerically) for a given wire radius r_w and circulation κ.

One may also consider alternatives to ^{4}He II. Apart from normal fluids mentioned before, Bose–Einstein condensates can exhibit a giant vortex [16] on the axis of rotation, where a central external repulsive potential plays the role of the wire. Such systems offer additional degree(s) of experimental access, but do require a separate derivation of the dispersion relations, for example regarding the form of the boundary condition at $r = r_w$. Based on our experience with different boundary conditions [4], we expect that bound states do also exist in such systems. The reported softening of dispersion relations for supersonic flows, and its traces in more realistic models of rotating super-fluids, require further investigation.

Let us finally remark that the family of bound states presented here is distinct from the phenomenon of Kelvin waves, known from normal and super-fluid dynamics. The position of the vortex considered in this paper is fixed, unable to move or be deformed – which is the important characteristic associated with Kelvin waves. Accordingly, the dispersion relation (e.g., in Fig. 2) of the modes considered here is much stiffer and more sound-like than that of the soft Kelvin waves with the dispersion $\omega(p_z) \sim \kappa p_z^2 \log(p_z)/2$ for small p_z, see [2].

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1. W. G. Unruh, Phys. Rev. Lett. **46**, 1351 (1981).
2. R.J. Donnelly, *Quantized Vortices in Helium II*, Cambridge University Press, 1991.
3. Ergoregion is a standard concept associated with stationary axisymmetric (i.e. rotating) spacetimes. Roughly speaking, all particles within the ergoregion are forced to co-rotate with the spacetime, as seen by non-rotating observers far away from the axis of rotation.
4. P. Marecki, J. Phys. A: Math. Theor. **45**, 295501 (2012).
5. A. L. Fetter, Phys. Rev. A **136**, 1488 (1964).
6. M. Stone, Phys. Rev. B **61**, 11780 (2000).
7. M. Flaig and U. R. Fischer, Phys. Rev. B **74**, 224503 (2006); U. R. Fischer and M. Visser, Phys. Rev. Lett. **88**, 110201 (2002).
8. M. Stone, Phys. Rev. E **62**, 1341 (2000); M. Stone, *Phonons and Forces: Momentum Versus Pseudomomentum in Moving Fluids*, in *Artificial Black Holes* (ed. by M. Novello, M. Visser, and G. Volovik), World Scientific, Singapore, 2002.
9. For $r_w > 1$, a Hamiltonian formulation of the problem exists [4] with the Hamiltonian H (different from $\mathcal{H}$) being a self-adjoint operator on a Hilbert space. Thus in this case all ω (eigenvalues of H) are real, and the family of corresponding modes is complete in the usual sense. For $r_w < 1$, however, the operator H is only a symmetric operator on a Krein space, and the presence of complex ω cannot be excluded *a priori*. Nonetheless, based on the (p_z, m, r_w)-dependence of real ω, we conjecture that they do not appear [4]. To the best of our knowledge, neither the problem of existence of complex frequencies, nor the issue of completeness of the eigenmodes of H has been settled in the literature for this case.
10. M. Richartz et al., Phys. Rev. D **80**, 124016 (2009).
11. S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time*, Cambridge University Press, 1989.
12. J. Brooks and R.J. Donnelly, J. Phys. Chem. Ref. Data **6**, 51 (1977).
13. S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability*, Dover Publications, 1981.
14. E. B. Sonin, Rev. Mod. Phys. **59**, 87 (1987).
15. G. E. Volovik, *The Universe in a Helium Droplet*, Calderon Press, Oxford, 2003.
16. U. R. Fischer and G. Baym, Phys. Rev. Lett. **90**, 140402 (2003); T. Isoshima and K. Machida, Phys. Rev. A **59**, 2203 (1999).