Pis'ma v ZhETF, vol. 97, iss. 4, pp. 195 – 196 © 2013 February 25

The first-order deviation of superpolynomial in an arbitrary representation from the special polynomial

A. Morozov[1)]

Moscow State University, 119991 Moscow, Russia

Institute for Theoretical and Experimental Physics, 117218 Moscow, Russia

Submitted 22 November 2012
Resubmitted 25 January 2013

Like all other knot polynomials, the superpolynomials should be defined in arbitrary representation R of the gauge group in (refined) Chern–Simons theory. However, not a single example is yet known of a superpolynomial beyond symmetric or antisymmetric representations. Following the article *Equations on knot polynomials and 3d/5d duality*, we consider the expansion of the superpolynomial around the special polynomial in powers of $q-1$ and $t-1$ and suggest a simple formula for the first-order deviation, which is presumably valid for arbitrary representation. This formula can serve as a crucial lacking test of various formulas for non-trivial superpolynomials, which will appear in the literature in the near future.

DOI: 10.7868/S0370274X13040012

As it was shown in the article [1], some superpolynomials [2] in the case of symmetrical and antisymmetrical representations possess simple factorization properties in the case of $q=1$ and $t=1$ respectively, which extend the corresponding property of the ordinary special polynomial [3, 4]. Now this property was also checked in [5] for all twisted knots, but there are arguments, that it does not hold for some more complicated knots [6].

There can be three directions to continue this research:

– to look at the other knots,

– to see what happens if q or t deviate from unity: $q=1+\hbar+\ldots$ or $t=1+\bar{\hbar}+\ldots$

– and to look at arbitrary representations.

The third direction is most interesting, but the problem is that we do not have any examples of superpolynomials even in the case of representation $[2,1]$ and as a result we can not really check our conjectures. Thus we choose a way in between: try to imagine, what the answer could be for the infinitesimal deviation from special polynomials, but for arbitrary representation and, perhaps, for generic knots.

Let us parameterize the small deviations of q and t from unity as follows:

$$q=e^{\hbar}; \quad t=e^{\overline{\hbar}}. \tag{1}$$

In this parametrization our superpolynomial can be written as $P_R(A,\hbar,\overline{\hbar})$. The special polynomial arises at $\hbar=0$ or $\overline{\hbar}=0$, and it satisfies [4]

$$P_R(A,0,0)=H_R(A,0)=\left[H_{\square}(A,0)\right]^{|R|}, \tag{2}$$

where $\square$ denotes the fundamental representation, and $|R|$ is the number of boxes in the Young Diagram of representation R.

In the next approximation we have:

$$P_R(A,\hbar,\overline{\hbar})=\sigma_{\square}^{|R|}(A)+\hbar\eta_R(A)+\overline{\hbar}\overline{\eta}_R(A)+\ldots, \tag{3}$$

where $\sigma_{\square}=H_{\square}(A,0)$, and $\eta_R=\left.\frac{\partial[P_R(A,\hbar,0)]}{\partial\hbar}\right|_{\hbar=0}$, $\overline{\eta}_R=\left.\frac{\partial\left[P_R(A,0,\overline{\hbar})\right]}{\partial\overline{\hbar}}\right|_{\bar{\hbar}=0}$.

Now let us see what can be said about the functions η and $\bar{\eta}$. For symmetric representations we use the factrorization property [1] of the special *super*polynomial (assuming that it is true for our knot):

$$P_{[r]}(A,0,\bar{\hbar})=\Big[P_{\square}(A,0,\bar{\hbar})\Big]^r. \tag{4}$$

This relation is conjecturally true for all $\bar{\hbar}$, but we need it only in the first order, when it implies:

$$\overline{\eta}_{[r]}(A)=r\sigma_{\square}^{r-1}(A)\overline{\eta}_{\square}(A). \tag{5}$$

Similarly [1], for antisymmetric representation

$$P_{[1^r]}(A,\hbar,0)=\Big[P_{\square}(A,\hbar,0)\Big]^r \tag{6}$$

implies

$$\eta_{1^r}(A)=r\sigma_{\square}^{r-1}(A)\eta_{\square}(A). \tag{7}$$

[1)]e-mail: anton.morozov@itep.ru

Namely, in the HOMFLY case, when ($\hbar = \bar\hbar$) we have

$$\eta_R(A) + \bar\eta_R(A) = \varkappa_R \sigma_\square^{|R|-2}\sigma_2(A), \tag{8}$$

where $\varkappa_R = \nu_{\overline{R}} - \nu_R$ and $\nu_R = \sum_i r_i(i-1)$. Here r_i is a height of the column number i in the Young diagram of the representation R. Finally, $\sigma_2(A)$ is the second special polynomial, like $\sigma_\square(A)$ it depends on the knot.

It is instructive to see how the reflection symmetry acts on the η-functions. According to [4] (see also [6] and references therein), it interchanges $R \leftrightarrow \overline{R}$ and also $q \leftrightarrow 1/t$. Then $\hbar \leftrightarrow -\bar\hbar$ and the symmetry implies that

$$\begin{aligned} \eta_R &\to -\bar\eta_{\overline{R}}, \\ \bar\eta_R &\to -\eta_{\overline{R}}. \end{aligned} \tag{9}$$

Since for the fundamental representation $\square = \overline{\square}$, we see that $\overline{\eta_\square} = -\eta_\square$ and

$$\begin{aligned} P_\square = \sigma_\square + \hbar\eta_\square + \bar\hbar\bar\eta_\square + \ldots = \sigma_\square + \hbar\eta_\square - \\ - \bar\hbar\eta_\square + \ldots = \sigma_\square - \hbar\bar\eta_\square + \bar\hbar\bar\eta_\square + \ldots . \end{aligned} \tag{10}$$

Now we can summarize what we know about symmetric and antisymmetric representations. From (3) we have

$$P_{[r]} = \sigma_{[r]} + \hbar\eta_{[r]} + \bar\hbar\bar\eta_{[r]} + \ldots \tag{11}$$

while from (8) we know that

$$\eta_{[r]} = \varkappa_R \sigma_\square^{|R|-2}\sigma_2 - \bar\eta_{[r]}. \tag{12}$$

Substituting one into another, we get:

$$\begin{aligned} P_{[r]} = \sigma_\square^r + \bar\eta_{[r]}(\bar\hbar - \hbar) + \hbar\sigma_\square^{r-2}\sigma_2\varkappa_{[r]} + \ldots = \\ = P_\square^r + \hbar\varkappa_{[r]}\sigma_\square^{r-2}\sigma_2 + \ldots . \end{aligned} \tag{13}$$

Similarly, for the antisymmetrical case:

$$\begin{aligned} P_{[1^r]} = \sigma_\square^r + \eta_{[1^r]}(\hbar - \bar\hbar) - \bar\hbar\varkappa_{[1^r]}\sigma_\square^{r-2}\sigma_2 + \ldots = \\ = P_\square^r - \bar\hbar\varkappa_{[1^r]}\sigma_\square^{r-2}\sigma_2 + \ldots . \end{aligned} \tag{14}$$

Now, let us compare these two formulas. They differ: one contains $\hbar$ and another $\bar\hbar$, but now we can observe that for symmetric and antisymmetric representations $\varkappa_R$ is rather special: since $\nu_{[r]} = 0$, $\varkappa_{[r]} = -\varkappa_{[1^r]} = \nu_{[1^r]} = \nu_{\overline{[r]}}$ and both formulas can be rewritten in a unified form:

$$P_R = P_\square^{|R|} + (\hbar\nu_{\overline{R}} - \bar\hbar\nu_R)\,\sigma_\square^{|R|-2}\sigma_2 + \ldots \tag{15}$$

This is a remarkable formula, because in this form it can be used for *arbitrary* representation, not obligatory symmetric and antisymmetric. This formula is our new conjecture for the first deviation of arbitrary superpolynomial from the special one. At the moment there is no way to test this formula, because nothing is known yet about the superpolynomials beyond (anti)symmetric representations. For the first attempt to make use of (16) – in the case of the figure-eight knot and $R = [2,1]$, and with a somewhat controversial result – see [7]. Further advances in this direction are very desirable.

I am indebted to Alexei Morozov for useful discussions. My work is partly supported by the Ministry of Education and Science of the Russian Federation under the contract 8498, by the grants NSh-3349.2012.2, RFBR-12-01-00525.

1. A. Morozov, arXiv:1208.3544.
2. N. M. Dunfield, S. Gukov, and J. Rasmussen, Experimental Math. **15**, 129159 (2006); math/0505662.
3. K. Liu and P. Peng, arXiv:0704.1526; Sh. Zhu, arXiv:1206.5886.
4. P. Dunin-Barkovsky, A. Mironov, A. Morozov et al., arXiv:1106.4305.
5. S. Nawata, P. Ramadevi, Zodinmawia, and X. Sun, arXiv:1209.1409.
6. S. Gukov and M. Stosic, arXiv:1112.0030
7. A. Anokhina, A. Mironov, A. Morozov, and A. Morozov, arXiv:1211.6375