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Tkachenko waves
In memory of V.K. Tkachenko

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This is a short review of theoretical and experimental studies of Tkachenko waves starting from their theoretical prediction by Tkachenko about 50 years ago up to their unambiguous experimental observation in the Bose–Einstein condensate of cold atoms.

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1. Introduction. In the year 2013 Vladimir Konstantinovich Tkachenko passed away in the age of 76. This sad event urges to review the legacy of this brilliant scientist in physics.

His active life in physics unfortunately was very short because of health problems. He published not more than about 10 papers but what papers! Tkachenko, being nominally (and really) an experimentalist, published the papers, which Dyson [1] called "*a tour de force* of powerful mathematics". Tkachenko's seminal works on a vortex lattice in superfluid helium and its oscillation were written about 50 years ago but up to now they remain actual and challenging in various areas of physics, superfluid liquids, cold-atom Bose–Einstein condensates, and astrophysics among them.

The series of Tkachenko's papers on dynamics of vortex lattices started from the paper [2], in which he calculated exactly the energy of an arbitrary periodic vortex lattice and showed that the triangular lattice has the lowest energy as in the mixed state of type II superconductors. In the second paper [3] he found (also exactly) the spectrum of waves in the vortex lattice for all wave vectors in the Brillouin zone. These waves are now called Tkachenko waves. Finally in his third paper [4] he demonstrated that in the long-wavelength limit the Tkachenko wave is nothing else as a transverse sound wave in the vortex lattice and its frequency is determined by the shear elastic modulus.

The following short review addresses the original theory of Tkachenko waves suggested for superfluid ^{4}He and its nowadays extension on Tkachenko waves in the Bose–Einstein condensate of cold atoms, and also overviews a long and controversial story of attempts to detect Tkachenko waves experimentally first in liquid ^{4}He and pulsars and then in Bose–Einstein cold-atom condensates, which culminated in an unambiguous observation of Tkachenko waves.

2. Tkachenko waves from the elasticity theory of a two-dimensional vortex crystal. We start not from the exact solution but from a more transparent approach deriving the Tkachenko wave from the elasticity theory of the vortex lattice.

The equation of motion in the continuous elasticity theory for atoms in the crystal lattice is the second Newton law:

$$\rho \frac{d^2\mathbf{u}}{dt^2} = \mathbf{f}, \quad (1)$$

where ρ is the mass density, $\mathbf{u}$ is the atom displacement, and the force $\mathbf{f}$ is defined as a functional derivative of the elastic energy of the crystal:

$$\mathbf{f} = -\frac{\delta E}{\delta \mathbf{u}} = -\frac{\partial E}{\partial \mathbf{u}} + \nabla_i \left(\frac{\partial E}{\partial \nabla_i \mathbf{u}}\right) = \nabla_i \left(\frac{\partial E}{\partial \nabla_i \mathbf{u}}\right). \quad (2)$$

We took into account translational invariance, which eliminates the dependence of the energy from the constant displacement $\mathbf{u}$.

Like in the elasticity theory, in vortex dynamics one can also introduce a continuous medium approximately describing an array of discrete vortex lines. This means that one carries out averaging (coarse-graining) of the equations of hydrodynamics over rather long scales of the order of intervortex distance. The approach is accurate enough as far as parameters of the medium do not vary essentially at the intervortex distance. This approach was called in Ref. [5] *macroscopic hydrodynamics*. In contrast to the elasticity theory of atomic crystals, the equation of vortex motion connects the force on the vortex not with an acceleration but with velocities:

$$-\rho \cdot 2\boldsymbol{\Omega} \times (\mathbf{v}_L - \mathbf{v}) = \mathbf{f}, \quad (3)$$

where $\boldsymbol{\Omega}$ is the angular velocity vector, $\mathbf{v}_L = d\mathbf{u}/dt$ is the vortex velocity, and $\mathbf{v}$ is the average velocity of the

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liquid. We consider the $T = 0$ case when the center-of-mass velocity coincides with the superfluid velocity. The angular velocity Ω determines the vortex density $n_v = 2\Omega/\kappa$, where $\kappa = h/m$ is the circulation quantum and m is the mass of a particle. The forces in Eq. (3) are forces on all vortices piercing a unit area. In classical hydrodynamics the left-hand side of Eq. (3) is called *Magnus force*. But in the theory of superfluidity and superconductivity they usually relate the Magnus force only with the term proportional to the vortex velocity $\mathbf{v}_L$, while the term proportional to the fluid current $\rho\mathbf{v}$ is called *Lorentz force*.

The equations for vortex displacements must be supplemented by the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{v}) = 0, \tag{4}$$

and by the Euler equation, which in the rotating coordinate frame is [5]

$$\frac{\partial\mathbf{v}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{v}_L = -\boldsymbol{\nabla}\mu. \tag{5}$$

The continuity and the Euler equations allow to determine the liquid velocity $\mathbf{v}$ and the chemical potential μ.

The expression for the elastic force can be obtained on the phenomenological basis taking into account hexagonal symmetry of the triangular lattice. We consider a 2D problem in the xy plane normal to the angular velocity vector $\boldsymbol{\Omega}$ (the axis z) with no dependence on z. The general expression for the elastic energy density in the 2D case is [6]

$$E_{\rm el} = \frac{C_{11}}{2}(\boldsymbol{\nabla}\cdot\mathbf{u})^2 + \\ + \frac{C_{66}}{2}\left[(\nabla_y u_x + \nabla_x u_y)^2 - 4\nabla_x u_x\nabla_y u_y\right]. \tag{6}$$

Here C_{11} is the inplane compressibility modulus and C_{66} is the shear modulus. We used here Voigt's notations for elastic moduli [7] adopted in the theory of superconductivity. Equation (6) is a particular case of a more general expression given in Refs. [5, 8], which took into account the z dependence. From Eqs. (2) and (6) one obtains an expression for the force on vortices:

$$\mathbf{f} = (C_{11} - C_{66})\nabla(\boldsymbol{\nabla}\cdot\mathbf{u}) + C_{66}\Delta\mathbf{u}. \tag{7}$$

The term proportional to the divergence $\boldsymbol{\nabla}\cdot\mathbf{u}$ can be neglected in the low frequency (long wavelength) limit. Then the components of the force $f_i = -\nabla_j\sigma_{ij}$ are determined by the stress tensor

$$\sigma_{ij} = -C_{66}(\nabla_i u_j + \nabla_j u_i) \tag{8}$$

for purely shear deformation. Here subscripts i and j take only two values x and y corresponding to the two axes in the xy plane. Then the Eq. (3) of vortex motion becomes

$$\frac{\partial\mathbf{u}}{\partial t} = \mathbf{v}_L = \mathbf{v} + \frac{C_{66}}{2\Omega\rho}[\hat{z}\times\Delta\mathbf{u}]. \tag{9}$$

It is convenient to divide the vortex displacement field $\mathbf{u}(\mathbf{r})$ and the fluid velocity field $\mathbf{v}(\mathbf{r})$ into longitudinal and transverse parts ($\mathbf{u} = \mathbf{u}_\parallel + \mathbf{u}_\perp$, $\mathbf{v} = \mathbf{v}_\parallel + \mathbf{v}_\perp$), so that $\boldsymbol{\nabla}\cdot\mathbf{u}_\perp = \boldsymbol{\nabla}\cdot\mathbf{v}_\perp = 0$ and $\boldsymbol{\nabla}\times\mathbf{u}_\parallel = \boldsymbol{\nabla}\times\mathbf{v}_\parallel = 0$. In an incompressible liquid $\mathbf{v}_\parallel = 0$ and the liquid velocity $\mathbf{v} = \mathbf{v}_\perp$ is purely transverse. Then Eq. (5) after integration over time yields

$$\mathbf{v} = -[2\boldsymbol{\Omega}\times\mathbf{u}_\parallel]. \tag{10}$$

After exclusion of $\mathbf{v}$ Eq. (9) yields the equations for longitudinal and transverse displacements $\mathbf{u}_\parallel$ and $\mathbf{u}_\perp$:

$$\frac{\partial\mathbf{u}_\parallel}{\partial t} = \frac{C_{66}}{2\Omega\rho}[\hat{z}\times\Delta\mathbf{u}_\perp], \tag{11}$$

$$\frac{\partial\mathbf{u}_\perp}{\partial t} = -2\boldsymbol{\Omega}\times\mathbf{u}_\parallel. \tag{12}$$

Excluding the small longitudinal displacement $\mathbf{u}_\parallel$ from equations one obtains an equation similar to that for the transverse sound in the conventional elasticity theory:

$$\frac{\partial^2\mathbf{u}_\perp}{\partial t^2} = c_{\rm T}^2\Delta\mathbf{u}_\perp, \tag{13}$$

with

$$c_{\rm T} = \sqrt{\frac{C_{66}}{\rho}} \tag{14}$$

being the velocity of the Tkachenko wave $\propto e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$ with the sound-like spectrum $\omega = c_{\rm T}k$. Vortices in the Tkachenko wave move on elliptical paths, but the longitudinal component $\mathbf{u}_\parallel$ parallel to the wave vector $\mathbf{k}$ is proportional to a small factor ω/Ω (see Eq. (12)). Thus it is fairly accurate to consider the Tkachenko wave to be a transverse sound wave in the two-dimensional lattice of rectilinear vortices [4]. Comparing Eqs. (10) and (12) one can see that in our approximation the liquid and the vortices move with the same velocity. The phenomenological approach cannot provide the value of the shear modulus. But it is clear that the elastic energy is in fact the kinetic energy of the velocity field induced by the vortices, and scaling estimations show that the shear modulus should be on the order of $C_{66} \sim \rho\kappa\Omega$. Its exact value can be obtained from the exact value of the energy of the vortex lattice obtained by Tkachenko (Sec. 3).

All experiments on Tkachenko waves dealt with finite cylindric liquid samples, and it is necessary to know the boundary conditions for Tkachenko cylindric waves. We restrict ourselves with axisymmetric modes.

The field of transverse displacements $\mathbf{u}_\perp$ may be determined by a vector potential $\boldsymbol{\Psi} = \Psi\hat{z}$:

$$\mathbf{u}_\perp = \boldsymbol{\nabla} \times \boldsymbol{\Psi} = -\hat{z} \times \boldsymbol{\nabla}\Psi. \qquad (15)$$

The potential Ψ must satisfy the wave equation

$$\frac{\partial^2\Psi}{\partial t^2} - c_{\rm T}^2 \Delta\Psi = 0. \qquad (16)$$

Axisymmetric modes with the sound-like spectrum $\omega = c_T k$ correspond to a cylindrical wave

$$\Psi = \Psi_0 J_0(kr) e^{-i\omega t},$$
$$u_r \approx 0, \quad u_\varphi = -\frac{\partial\Psi}{\partial r} = k\Psi_0 J_1(kr) e^{-i\omega t}, \qquad (17)$$

where subscripts r and φ denote radial and azimuthal components in the cylindric coordinate frame (r,φ).

Suppose that no external force acts upon the liquid, which fills a cylinder of the radius R. Then eigenfrequencies are defined by the condition that the total angular momentum M does not vary. Since in the Tkachenko wave the fluid and vortices move together

$$M = 2\pi\rho \int_0^R v_\varphi r^2 dr = -2i\omega\pi\rho \int_0^R u_\varphi r^2 dr = \\ = -2i\omega\pi\rho\Psi_0 R^2 J_2(kR) e^{-i\omega t}. \qquad (18)$$

The condition $M = 0$ yields eigenfrequencies

$$\omega_i = j_{2,i}\frac{c_{\rm T}}{R}, \qquad (19)$$

where $j_{2,i}$ denotes the ith zero of the Bessel function $J_2(z)$. For the fundamental frequency $j_{2,1} = 5.14$. This is a result obtained by Ruderman [9] who discussed Tkachenko modes in pulsars (see Sec. 4).

The condition $M = 0$ is equivalent to the boundary condition that the azimuthal component of the momentum flux through the liquid boundary $r = R$ vanishes. This momentum flux is given by the relevant stress tensor component $\sigma_{\varphi r}$ in cylindrical coordinates:

$$\sigma_{\varphi r}(r) = -\rho c_{\rm T}^2 \left(\frac{\partial u_\varphi(r)}{\partial r} - \frac{u_\varphi(r)}{r}\right). \qquad (20)$$

The condition $\sigma_{\varphi r}(R) = 0$ requires that

$$\frac{\partial u_\varphi(R)}{\partial r} - \frac{u_\varphi(R)}{R} = 0. \qquad (21)$$

This yields the same spectrum Eq. (19) as the condition $M = 0$.

3. Exact solution of Tkachenko. Tkachenko has found an exact solution for the vortex lattice and its oscillation using the theory of elliptic functions on the complex plane [2, 3]. It is well known that a two-dimensional vector $\mathbf{r}(x,y)$ can be presented as a complex variable $z = x + iy$. Then the velocity field $v(z) = v_x + iv_y$ induced by vortices located in nodes of a vortex lattice with position vectors $z_{kl} = 2k\omega_1 + 2l\omega_2$ (k and l are arbitrary integers) is given by

$$v(z) = \frac{\kappa}{2\pi}[\zeta^*(z) - \lambda z^*], \qquad (22)$$

where λ is a constant, which will be defined below, and

$$\zeta(z) = \frac{1}{z} + \sum_{k,l}{}' \left(\frac{1}{z - z_{kl}} + \frac{1}{z_{kl}} + \frac{z}{z_{kl}^2}\right) \qquad (23)$$

is the quasiperiodic Weierstrass zeta function [10] with two complex semi-periods ω_1 and ω_2 and a prime means exclusion of the term $k = l = 0$ from the sum. The quasiperiodicity conditions are

$$\zeta(z + 2k\omega_1) = \zeta(z) + 2k\omega_1,$$
$$\zeta(z + 2l\omega_2) = \zeta(z) + 2l\omega_2. \qquad (24)$$

The lattice is shown in Fig. 1 for the semi-periods $\omega_1 = a/2$ and $\omega_2 = be^{i\alpha}/2$. The unit cell area of the lattice is

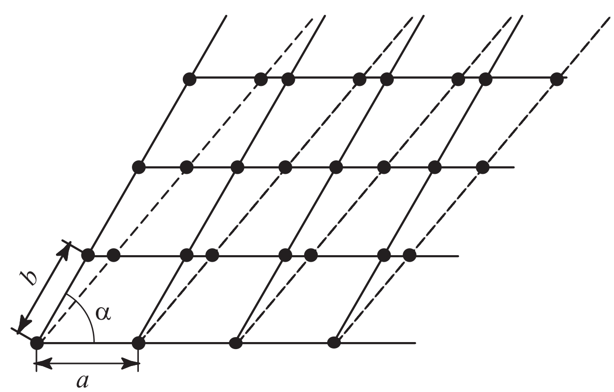

Fig. 1. Vortex lattice before (solid lines) and after (dashed lines) shear deformation

$$A = 4\mathrm{Im}(\omega_1^*\omega_2) = ab\sin\alpha. \qquad (25)$$

Tkachenko has shown that a lattice with arbitrary semi-periods rotates as a solid body with the angular velocity $\Omega = \kappa/2A$, if one chooses λ satisfying the condition

$$\zeta(\omega_1) + \lambda\omega_1 = \Omega\omega_1^*. \qquad (26)$$

Taking into account the exact relation for Weierstrass zeta function,

$$\omega_2\zeta(\omega_1) - \omega_1\zeta(\omega_2) = \frac{i\pi}{2}, \qquad (27)$$

another condition necessary for solid-body rotation is also satisfied:

$$\zeta(\omega_2) + \lambda\omega_2 = \Omega\omega_2^*. \tag{28}$$

For symmetric triangular and quadratic lattices $\lambda = 0$.

The velocity field being known Tkachenko [2] found after ingenious manipulations with integrals over elliptic functions the exact value of kinetic energy per unit area in the rotating coordinate frame for an arbitrary vortex lattice:

$$E = \frac{\rho\kappa\Omega}{2\pi}\left[\ln\frac{2|\omega_1\omega_2|^{1/2}}{\pi r_c} - \frac{\ln 2}{3}\left|\theta_1'(0,\tau)\theta_1'\left(0,-\frac{1}{\tau}\right)\right|\right] = \frac{\rho\kappa\Omega}{2\pi}\left[\ln\frac{\sqrt{A|\tau|}}{\pi r_c\sqrt{\tau_I}} - \frac{\ln 2}{3}\left|\theta_1'(0,\tau)\theta_1'\left(0,-\frac{1}{\tau}\right)\right|\right], \tag{29}$$

where the complex parameter

$$\tau = \tau_R + i\tau_I = \omega_2/\omega_1 = \frac{b}{a}e^{i\alpha} \tag{30}$$

determines the type of the lattice,

$$\theta_1(z,q) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2} e^{i(2n+1)z} \tag{31}$$

is one from the elliptic theta functions, and $\theta_1'(z,q)$ is its derivative with respect to the first argument z. The energy has a minimum at $\tau = e^{i\pi/3}$ ($a = b = \sqrt{\kappa/\sqrt{3}\Omega}$, $\alpha = \pi/3$), which corresponds to the triangular lattice with the energy density

$$E_m = \frac{\rho\kappa\Omega}{2\pi}\left(\ln\frac{\sqrt{A}}{r_c} - 1.321\right).$$

In order to find the shear modulus let us deform the triangular lattice without varying the vortex density as shown in Fig. 1. Then only the real part of τ varies proportionally to the shear deformation $u_{xy} = \frac{1}{2}(\nabla_y u_x + \nabla_x u_y)$: $\delta\tau = \delta\tau_R = 2u_{xy}\sin\alpha$. Expanding the energy density Eq. (29) with respect to τ_R and comparing it with the elastic energy (6) with $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ one obtains the exact value of the shear modulus:

$$C_{66} = \rho c_{\mathrm{T}}^2 = \frac{\rho\kappa\Omega}{8\pi}. \tag{32}$$

4. Tkachenko waves in pulsars. Some features of pulsar behavior have been explained by the hypothesis that the rotating inner matter of pulsars is in the superfluid state and is threaded by vortex lines. Among such features were sudden spin-ups of pulsars (glitches) and slow relaxation after the glitch [1, 11, 12]. In addition, very slow oscillations of the Crab pulsar's period have been observed [1]. Ruderman [9] has associated this remarkable phenomenon with Tkachenko waves. He considered waves in a cylinder, ignoring the difference between cylindrical and spherical geometry. Inserting into Eq. (19) the data for the pulsar in the Crab nebula ($\Omega = 200\,\mathrm{rad/s}$, $R = 10^6\,\mathrm{cm}$, $\kappa = 2\cdot 10\,\mathrm{cm^2/s}$), Ruderman found that the oscillation period for the fundamental mode $s = 1$ should be

$$T = \frac{2\pi}{\omega_1} = 9.73\cdot 10^6\ \mathrm{s} = 3.75\ \mathrm{months}$$

in good agreement with the observed period of ~ 3 months. Dyson [1] argued that it is difficult to think of any other internal motion, which would have a time scale as long as this.

Ruderman's model was rather idealized even for very long cylinders when pinning of vortices to the solid surface is important. In pulsars the solid crust confining the neutron matter plays the role of a solid surface. More on this issue was discussed in Ref. [5].

Later the interest to interpretation of pulsar oscillations in the terms of the Tkachenko mode declined to some extent and other interpretations were suggested. But recently some publications urged to return to the Tkachenko-mode interpretation of long-period pulsar oscillations [13–15].

5. Search of Tkachenko waves in superfluid ^{4}He. The first attempt to observe a Tkachenko wave in a laboratory was undertaken by Tkachenko himself in the 1970s in a study of torsion oscillations of a light cylinder immersed into rotating superfluid ^{4}He and suspended by a thin fiber [16]. The oscillating cylinder cannot drag the superfluid component of the liquid but it does drag the normal one. The latter makes superfluid vortices to oscillate via mutual friction. No conclusive data were obtained in this experiment. A later analysis of this case (see Sec. VIII.D in Ref. [5]) showed that an essential contribution of the Tkachenko mode would be possible for rather fast rotation inaccessible at that time.

Further efforts to discover Tkachenko waves experimentally were stimulated by Ruderman's theory explaining long-period oscillations of the pulsar rotation velocity. For simulation of the process in pulsars, J. Tsakadze and S. Tsakadze [17, 18] studied free rotation of buckets of various shapes, cylindrical included, filled with He II, and revealed rotation-period oscillations superimposed on the steady deceleration of rotation. The oscillations disappeared above the λ point that proved their superfluid nature. But the oscillation frequencies observed for cylindrical vessels were nearly eight times higher than the fundamental frequency predicted by Ruderman [9] for this geometry. This disagreement was explained by three-dimensional effects of pinning and bending of vortices [19, 5]. These effects transform the Tkachenko mode into a mixed mode combining the Tkachenko wave and the inertial wave with the spectrum [19, 20]

$$\omega = \sqrt{4\Omega^2 \frac{p^2}{k^2+p^2} + c_{\rm T}^2 k^2}. \tag{33}$$

Here p is the z component of the wave vector, which appears in the mixed plane wave $\propto e^{i\mathbf{k}\cdot\mathbf{r}+ipz-i\omega t}$ as a result of pinning at surfaces normal to the rotation axis (the axis z). Without the quantum contribution $c_{\rm T}^2 k^2$ this is an inertial wave well known in hydrodynamics of rotating classical fluids [21]. The quantum contribution depends on the container radius R since for the Tkachenko-wave resonance $k \sim 1/R$. The Tkachenko velocity usually is very small (of order 1 cm/s). As a result, the frequency $\omega = c_{\rm T} k$ of the pure Tkachenko mode is much smaller than Ω. Then according to Eq. (33) even rather weak pinning leading to rather weak vortex bending (small p) can strongly influence the mode frequency. As a result, the inertial-wave contribution essentially exceeds the quantum (Tkachenko) contribution. The inertial-wave contribution grows with decreasing of the height L of helium in the container (the length of vortices). J. Tsakadze and S. Tsakadze [17, 18] used cylindric containers of moderate aspect ratio L/R when the quantum Tkachenko contribution was negligible. Therefore they observed the inertial-wave resonance. This was proven by experimental detection [22] of properties predicted by the theory of the initial-wave resonance [19]. The observed oscillation frequency depended on L and on the smoothness of the bottom, but did not depend on the container radius R (see more detailed comparison and discussion in Refs. [5, 22, 23]).

In further experiments, S. Tsakadze [24] used longer cylindrical containers in an effort to reach the conditions when pure Tkachenko waves are possible. He could not do it completely, because it required impractical containers with too large ratios L/R, but he managed to come fairly close to the case when the Tkachenko contribution to the oscillation frequency was of the same order as the inertial-wave contribution. S. Tsakadze noticed an essential deviation from the frequency of the inertial wave resonance. The deviation roughly agreed with what was expected from the mixed-wave resonance when the classical and the quantum contributions to the spectrum Eq. (33) were of the same order. This was the first experimental evidence of the Tkachenko elasticity and consequently of crystalline order in the vortex lattice.

The next attempt of observation of the Tkachenko wave was undertaken by Andereck et al. [25, 26], who claimed that they saw Tkachenko waves in the experiment on torsional oscillations of a pile-of-disks immersed into a rotating superfluid ^{4}He. They observed a resonance, which they connected with a peak in the density of state caused by a minimum of the spectrum Eq. (33) at given p. But the theoretical analysis of Andereck et al. left unresolved a serious problem (by admission of the authors themselves; see p. 288 in the paper by Andereck et al. [26]): how can the oscillations of disks, introducing perturbations with wavelengths of the order of the disk radius, generate waves whose wavelengths are an order of magnitude smaller than the radius of the disks? Andereck et al. believed that they observed Tkachenko modes for very low aspect ratio L/R (L in their case was a small distance between disks), which was in conflict with the conclusion that because of pinning observation of Tkachenko modes requires very high aspect ratio. Later it was demonstrated [27] (see also Ref. [5]) that the resonance observed by Andereck et al. can be readily explained as a predominantly inertial-wave resonance without any contribution of Tkachenko rigidity.

In summary, experimental observation of Tkachenko waves in superfluid ^{4}He encountered serious problems connected with pinning of vortices at solid surfaces containing superfluids. Evidences of the Tkachenko mode were rather circumstantial and did not allow a decisive quantitative comparison with the theory. A breakthrough in experimental observation of Tkachenko waves became possible after discovery of a new type of superfluids: Bose–Einstein condensates of cold atoms. In these new superfluids a superfluid sample being confined by a potential trap has no contacts with any solid surface. This excludes the main hurdles for observation of pure Tkachenko waves: pinning and competition with the inertial-wave resonance. However, a number of assumptions used in the Tkachenko theory became invalid: incompressibility and homogeneity of the liquid. This required revision of the theory, which will be discussed in the following sections.

6. Tkachenko wave in a compressible perfect fluid. For a discussion of the effect of compressibility on vortex oscillations we need to return to the general linear equations of motion, Eqs. (4), (5), and (9). Let us neglect first inhomogeneity of the liquid. Then the equations of motion have a plane wave solution $\propto e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t}$ and after linearization become

$$-i\omega\rho' + \rho\mathbf{k}\cdot\mathbf{v} = 0, \tag{34}$$

$$-i\omega\mathbf{v} + 2\boldsymbol{\Omega}\times\mathbf{v}_L + i\mathbf{k}\mu' = 0, \tag{35}$$

$$-i\omega\mathbf{u} = \mathbf{v} - \frac{C_{66}k^2}{2\Omega\rho}[\hat{z}\times\mathbf{u}], \qquad (36)$$

where ρ' and μ' are small corrections to the mass density ρ and the chemical potential μ induced by the propagating wave. Using the relation $\mu' = (c_s^2/\rho)\rho'$ where c_s is the sound velocity one can exclude the density correction ρ' and obtain the equation connecting the velocities $\mathbf{v}$ and $\mathbf{v}_L$:

$$-i\omega\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v}_{\mathrm{L}} + \frac{c_s^2}{\rho}i\mathbf{k}\frac{\mathbf{k}\cdot\mathbf{v}}{i\omega} = 0. \qquad (37)$$

We should solve two 2D vector Eqs. (36) and (37). One can divide the velocity and the displacement fields into longitudinal (parallel to $\mathbf{k}$) and transverse (perpendicular to $\mathbf{k}$) parts again. We consider only low frequencies $\omega \ll c_s k$ excluding usual sound waves. Then the transverse velocity $-i\omega\mathbf{u}_\perp$ approximately coincides with the transverse velocity $\mathbf{v}_\perp$ of the liquid and excluding small longitudinal velocity $-i\omega\mathbf{u}_\parallel$ Eqs. (36) and (37) reduce to two equations for the longitudinal and the transverse liquid velocities $\mathbf{v}_\parallel$ and $\mathbf{v}_\perp$:

$$\begin{gathered} 2\Omega i\omega v_\parallel = -\omega^2 v_\perp + \frac{C_{66}k^2}{\rho}v_\perp, \\ 2\Omega i\omega v_\perp = -c_s^2k^2 v_\parallel. \end{gathered} \qquad (38)$$

This yields the dispersion relation [19]

$$\omega^2 = \frac{c_s^2c_{\mathrm{T}}^2k^4}{c_s^2k^2 + 4\Omega^2}. \qquad (39)$$

This dispersion relation also follows from a more general expression obtained by Volovik and Dotsenko [28] using the method of Poisson brackets. In the limit $c_s \to \infty$ Eq. (39) yilds the spectrum of the Tkachenko wave in an incompressible liquid. The compressibility strongly alters the spectrum of this wave at small $k \ll 2\Omega/c_s$, making it parabolic:

$$\omega = \frac{c_s c_{\mathrm{T}}}{2\Omega}k^2. \qquad (40)$$

In superfluid ^{4}He and ^{3}He the effect of compressibility on inertial waves is rather academic, because the space scale c_s/Ω at which the incompressible-fluid hydrodynamics becomes invalid is extremely large (of order hundreds of meters) and is not relevant to any real laboratory experiment. So at that time, when this effect was first analyzed [19, 5], it was considered as a theoretical curiosity, or belonging to some astrophysical applications. The situation became essentially different after discovery of the BEC of cold atoms. In contrast to the both helium superfluids, BEC is a weakly interacting Bose gas with very low sound speed and very high compressibility. Importance of high liquid compressibility for Tkachenko waves in BEC was pointed out by Baym [29].

In a compressible liquid centrifugal forces make the liquid density essentially inhomogeneous at the scale c_s/Ω. At the distance $r \sim c_s/\Omega$ from the rotation axis the linear velocity Ωr of solid body rotation becomes of the same order as the sound velocity c_s. Therefore, the analysis of the homogeneous liquid presented above is purely illustrative and cannot be directly applied to practical problems arising in experiments of Tkachenko waves in BEC of cold atoms. One should take into account inhomogeneity of the fluid.

Let us now consider axisymmetric Tkachenko modes in a rotating finite BEC cloud of pancake geometry. One can ignore variations along the axis of the pancake, and the problem becomes two-dimensional. The 2D cloud is trapped by the parabolic potential $\frac{1}{2}m\omega_\perp^2 r^2$. The Thomas–Fermi approximation [30, 31] yields an inverted parabola distribution of the mass density: $\rho(r) = \rho(0)(1 - r^2/R^2)$. Here $R = \sqrt{2}c_s(0)/\sqrt{\omega_\perp^2 - \Omega^2}$ is the cloud radius (Thomas–Fermi radius) and $c_s(0)$ is the sound velocity at the symmetry axis $r = 0$. The radius R grows with the angular velocity because of the effect of centrifugal forces. The Tkachenko mode in such a geometry was investigated numerically with solving the equations of Gross–Pitaevskii theory [32, 33]. On the other hand, in experiments the cloud size R essentially exceeded the intervortex distance. Therefore a simpler approach based on the macroscopic hydrodynamics explained in Sec. 2 can provide a deeper insight into physics of the phenomenon [34].

Since compressibility effect becomes important at $k \sim \Omega/c_s$ and the eigenvalues of k are expected to be of the order of $1/R$ the compressibility effect is essential if the parameter

$$s = \frac{\Omega R}{\sqrt{2}c_s(0)} = \frac{\Omega}{\sqrt{\omega_\perp^2 - \Omega^2}} \qquad (41)$$

is of order of unity or more. Thus at rapid rotation of the BEC with angular velocity Ω close to the trap frequency $\omega_\perp$ liquid compressibility should be taken into account.

The equations of motion (38) for plane waves in a homogenous compressible liquid can be transformed to those describing a monochromatic axisymmetric cylindric mode $\propto e^{-i\omega t}$ in the cylindric system of coordinates:

$$2\Omega i\omega v_r = -\omega^2 v_t - \frac{1}{\rho(r)r^2}\frac{\partial}{\partial r}\left[\rho(r)c_{\mathrm{T}}^2r^3\frac{\partial}{\partial r}\left(\frac{v_t}{r}\right)\right], \qquad (42)$$

$$2\Omega i\omega v_t = \frac{\partial}{\partial r}\left\{\frac{c_s^2(r)}{\rho(r)r}\frac{\partial[\rho(r)rv_r]}{\partial r}\right\}. \qquad (43)$$

Longitudinal and transverse components correspond to radial (subscript r) and azimuthal (subscript φ) components respectively. Now ρ, c_s, and $C_{66} = \rho c_{\rm T}^2$ depend on the distance r from the rotation axis, but the Tkachenko velocity $c_{\rm T}$ does not depend on density. For a weakly interacting Bose gas c_s^2 is proportional to the density ρ. Therefore the ratio c_s^2/ρ is a constant equal to its value $c_s^2(0)/\rho(0)$ in the cloud center $r = 0$.

As well as in an incompressible liquid, the flux of the azimuthal component of the momentum through the liquid boundary $r = R$ given by Eq. (20) must vanish. Since the stress tensor (momentum flux) is proportional to ρ and the latter vanishes at $r = R$, it looks that the momentum flux through the boundary vanishes independently from whether the boundary condition Eq. (21) is satisfied or not. But this is not true. Solving the equation of motion (42) close to $r = R$ by expansion in small $(R-r)/R$ one obtains that $v_t \approx r[C_1 + C_2 \ln(R-r)]$, where C_1 and C_2 are arbitrary constants. The component $\propto C_2$ diverges at $r \to R$ and gives a finite contribution to the stress tensor despite the factor $\rho \propto R - r$. So this component should be absent. This requirement is satisfied only if the boundary condition Eq. (21) takes place.

In addition to the boundary condition (21) we need the second boundary condition imposed on radial liquid velocity v_r. We use the arguments similar to those used for derivation of Eq. (21). The total mass balance requires that the radial mass current $\rho(r)v_r(r)$ at the BEC cloud border $r = R$ vanishes. Solving Eq. (43) at $r \approx R$ by series expansion (again neglecting terms $\sim (R-r)^2$) one obtains:

$$v_r(r) = \frac{\Omega i\omega v_t(R) R}{c_s(0)^2}\frac{R-r}{2} + C_1\left(1 + \frac{R-r}{R}\right) + \frac{C_2}{R-r}. \qquad (44)$$

The divergent component $\propto C_2$ gives a finite mass flow at the border and should be eliminated. Taking a derivative from v_r and excluding the constant C_1 from the expressions for v_r and its derivative we receive the boundary condition imposed on v_r:

$$\frac{dv_r(R)}{dr} + \frac{v_r(R)}{R} = -\frac{i\omega\Omega R}{2c_s(0)^2}v_t(R). \qquad (45)$$

It is useful to introduce the dimensionless variables for Eqs. (42) and (43):

$$\tilde r = \frac{r}{R}, \quad \tilde\omega = \frac{\omega R}{c_{\rm T}}, \quad \tilde v_r = \frac{i v_r}{c_{\rm T}}, \quad \tilde v_t = \frac{v_t}{c_s(0)}. \qquad (46)$$

Then Eqs. (42) and (43) and the boundary conditions to them become purely real and depend only on the compressibility parameter s given by Eq. (41). Solving them numerically one obtains reduced eigenfrequencies $\tilde\omega_i = f_i(s)$ as functions of s. At large s the eigenfrequencies $\tilde\omega_i = \gamma_i/s$ are inversely proportional to s. The first two eigenfrequencies correspond to $\gamma_1 = 9.66$ and $\gamma_2 = 22.8$. Returning back to dimensional frequencies at rapid rotation ($\omega_\perp - \Omega \ll \omega_\perp$) their values are

$$\omega_i = \frac{\gamma_i}{s}\frac{c_{\rm T}}{R} \approx \sqrt{2}\gamma_i \frac{c_{\rm T}}{c_s(0)}(\omega_\perp - \Omega). \qquad (47)$$

Qualitatively this simple expression (aside from a numerical factor) follows from the dispersion relation (40) for Tkachenko plane waves taking into account that the eigenmodes of the cloud correspond to wave numbers $k \sim 1/R$.

Let us now address the experiment, which provided the first unambiguous experimental observation of Tkachenko waves. It is remarkable that in Bose–Einstein condensates of cold atoms it was possible to observe Tkachenko waves *visually*. Fig. 2 shows the image

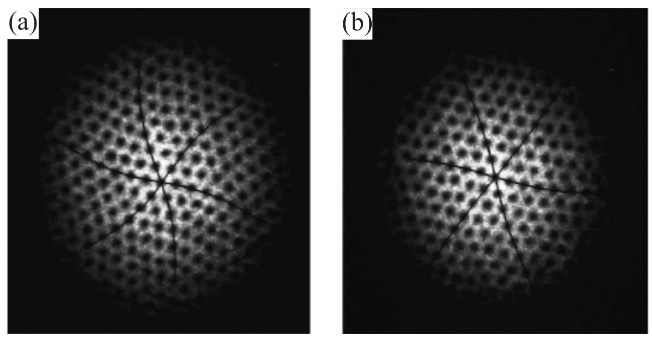

Fig. 2. Tkachenko mode excited in a rotating Bose–Einstein condensate of ^{87}Rb atoms by a pulse in two moments after the pulse [35]. Line are sin fits to distortions of the vortex lattice by the Tkachenko mode

of Tkachenko wave obtained by Coddington et al. [35] in a rotating Bose–Einstein condensate of ^{87}Rb atoms. In Fig. 3 black squares show experimental points [35] plotted in our dimensionless variables by I. Coddington. They were obtained for various parameters, but collapse on the same curve, as expected from the present analysis. The solid line in the same figure shows the numerically found first eigenfrequency ω_1 plotted as a function of $\Omega/\sqrt{\omega_\perp^2 - \Omega^2}$ (solid line). Quantitative agreement between the theory and the experiment looks quite good. Coddington et al. [35] measured also the ratio of the two first frequencies $\omega_2/\omega_1 = 1.8$ at $\Omega/\omega_\perp = 0.95$, which corresponds to $s = 3.04$. The present theory predicts the ratio $\omega_2/\omega_1 = 2.09$.

The agreement becomes worse at larger $s = \Omega/\sqrt{\omega_\perp^2 - \Omega^2}$. This can be connected with violation of the assumption, which the theory was based

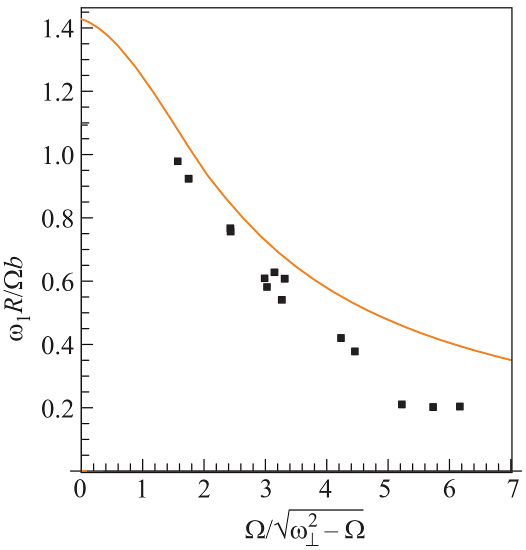

Fig. 3. Comparison between the theory (solid line) and the experiment (black squares). Here ω_1 is the first Tkachenko eigenfrequency and $b = \sqrt{\kappa/\sqrt{3}\Omega}$ is the intervortex distance

upon: the vortex lattice is an array of singular vortex lines with the core size (of the order of the coherence length $\xi \sim \kappa/c_s$) much less than the intervortex distance b. One can call this the Vortex Line Lattice (VLL) regime. With Ω approaching the trap frequency $\omega_\perp$ the cloud radius R grows, and if the total number of particles remain constant the particle density decreases. The sound velocity c_s decreases also and the core radius grows. When ξ becomes of the same order as the intervortex distance b the vortex cores start to overlap like in the mixed state of a type-II superconductor close to the second critical magnetic field $H_{c2} \sim \Phi_0/\xi^2$ (Φ_0 is the magnetic flux quantum). At the critical magnetic field H_{c2} the phase transition to the normal state takes place. However, in a rotating BEC there is no phase transition at the "critical" angular velocity $\Omega_{c2} \sim \kappa/\xi^2$. Instead the crossover to a new regime takes place: at $\Omega \gg \Omega_{c2}$ all atoms condense into a state, which is a coherent superposition of single-particle states in the Lowest Landau Level (LLL) similar to that of a charged particle in a magnetic field (the LLL regime). This interesting regime, which is called also the mean-field quantum Hall regime, is now the subject of intensive experimental and theoretical investigations [31].

7. Search of Tkachenko waves in the LLL regime. The plausible approach to the vortex dynamics in the LLL regime is that the phenomenological theory developed in Sec. 6 is still valid in this regime but one should revaluate elastic moduli and the expressions for the sound and the Tkachenko velocities c_s and c_{T}. For this revaluation we consider an infinite periodic vortex lattice in an infinite uniform liquid, neglecting first the trapping potential but taking into account interaction. In the Gross–Pitaevskii theory the Gibbs thermodynamic potential is

$$G = -m\mu|\psi|^2 + \frac{\hbar^2}{2m}\left|\left(-i\nabla - \frac{2\pi}{\kappa}\mathbf{v}_0\right)\psi\right|^2 + \frac{g}{2}|\psi|^4. \quad (48)$$

Here ψ is the BEC wave function, μ is the chemical potential, g is the interaction constant, and $\mathbf{v}_0 = [\boldsymbol{\Omega}\times\mathbf{r}]$ is the velocity of the solid body rotation. Let us consider the gauge transformation $\psi \to \psi e^{i\phi}$, $\mathbf{v}_0 \to \mathbf{v}_0 + (\kappa/2\pi)\boldsymbol{\nabla}\phi$ with constant $\boldsymbol{\nabla}\phi$. The Gibbs potential Eq. (48) is invariant with respect to this gauge transformation if it is accompanied by translation, which corresponds to a shift of the rotation axis.

The exact wave function for this state was found in the classical work by Abrikisov [36] for type-II superconductors close to H_{c2}, and later it was generalized for an arbitrary unit cell of the vortex lattice [37, 38]. As well as for type-II superconductors close to H_{c2}, in zero-order approximation one can neglect interaction (nonlinear term $\propto |\psi|^4$). Then the linear Schrödinger equation is similar to that for a charged particle in a uniform magnetic field:

$$m\mu\psi = -\frac{\hbar^2}{2m}\left[\left(\frac{\partial}{\partial x} - i\frac{2\pi v_{0x}}{\kappa}\right)^2 + \left(\frac{\partial}{\partial y} - i\frac{2\pi v_{0y}}{\kappa}\right)^2\right]\psi. \quad (49)$$

At $\mu = \hbar\Omega/m$ it has a solution, which corresponds to the lowest Landau level:

$$\psi_k \propto \exp\left[ikx - \frac{(y-y_k)^2}{2l^2}\right], \quad (50)$$

where $l^2 = \kappa/4\pi\Omega$ and $y_k = -l^2 k$. The solution is given for the gauge with $\mathbf{v}_0(-2\Omega y, 0)$. The frequency 2Ω is the analog of the cyclotron frequency $\omega_c = eH/mc$ for an electron in a magnetic field. If we consider a square $L\times L$ with periodic boundary conditions, then $k = -2\pi n/L$ with the integer n. Using the condition $0 < y_k < L$, one can see that the integer n should vary from zero to the integer closest to $L^2/2\pi l^2$. This is the total number of LLL states, which is exactly equal to the number of vortices $2\Omega L^2/\kappa$. All these states are orthogonal to each other and have the same energy. The degeneracy is lifted by taking into account the interaction energy. The solution, which corresponds to the periodic vortex lattice with one quantum per lattice unit cell, is [38]

$$\psi = \sum_n C_n \exp\left[inkx - \frac{(y + l^2 nk)^2}{2l^2}\right], \quad (51)$$

where $C_{n+1} = C_n \exp(2\pi i b \cos\alpha/a)$, a, b, and the angle α are the parameters of the unit lattice cell (see Fig. 1).

This solution yields the thermodynamic potential of the infinite BEC in the LLL regime averaged over the vortex lattice unit cell:

$$G = (-m\mu + \hbar\Omega)n + \frac{g}{2}\beta n^2, \tag{52}$$

where $n = \langle|\psi|^2\rangle$ is the average particle density and the parameter [38]

$$\beta = \frac{\langle|\psi|^4\rangle}{\langle|\psi|^2\rangle^2} = \sqrt{\tau_I}\left\{\left|\theta_3(0, e^{2\pi i\tau})\right|^2 + \left|\theta_2(0, e^{2\pi i\tau})\right|^2\right\} \tag{53}$$

depends on lattice parameters a, b, and α via the complex parameter τ determined by Eq. (30). Here

$$\theta_2(z, q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{i(2n+1)z}, \qquad \theta_3(z, q) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{i\cdot 2nz} \tag{54}$$

are theta functions [10]. The minimum of the interaction energy corresponds to the triangular vortex lattice with $\beta = 1.1596$, $a = b = 2l\sqrt{\pi/\sqrt{3}}$, $\alpha = \pi/3$. According to Eq. (52) the Gibbs potential has a minimum at the particle density $n = (m\mu - \hbar\Omega)/\beta g$. This allows to determine the sound velocity.

$$c_s = \sqrt{\rho\frac{\partial\mu}{\partial\rho}} = \sqrt{\frac{\beta g n}{m}}. \tag{55}$$

This insignificantly differs from the expression for the sound velocity in the VLL regime by the factor $\sqrt{\beta}$, which is very close to unity.

Calculation of the shear elastic modulus C_{66} in the LLL regime is similar to that in the VLL regime. Deforming the triangular lattice as shown in Fig. 1, the real part $\tau_{\rm R}$ of the complex parameter τ varies proportionally to the shear deformation u_{xy}. Expanding the expression Eq. (53) for β and comparing the term $\propto \delta\tau_{\rm R}^2$ in the thermodynamic potential, Eq. (52), with the elastic energy Eq. (6), one obtains the value of the shear modulus:

$$C_{66} = \frac{gn^2}{2}\frac{\partial^2\beta}{\partial\rho^2}\sin^2\alpha = 0.2054\rho c_s^2. \tag{56}$$

This agrees with the value of the shear modulus known [39, 40] for type-II superconductors close to the critical field H_{c2}[2)] and with the value obtained by Sinova et al. [41] (after taking into account the different definition of the elastic modulus c_{66} by Sinova et al.: $c_{66} = 2C_{66}$).

So in the LLL regime the Tkachenko velocity

$$c_T = \sqrt{\frac{C_{66}}{\rho}} = 0.453c_s \tag{57}$$

is of the same order as the sound velocity, in contrast to the VLL regime where $c_{\rm T} \sim c_s\xi/b$ is much smaller than c_s because of small ratio ξ/b.

Equations (47) and (57) yield a very simple expression for Tkachenko eigenfrequencies in the LLL regime:

$$\omega_i = 0.641\gamma_i(\omega_\perp - \Omega). \tag{58}$$

For the lowest eigenfrequency $i = 1$ with $\gamma_1 = 9.66$ (see the paragraph before Eq. (47)) this yields $\omega_1 = 6.19(\omega_\perp - \Omega)$. Note that the Tkachenko velocity in the LLL regime is smaller than its value $\sqrt{\kappa\Omega/8\pi}$ in the VLL regime because of small sound velocity c_s in the LLL regime. However, the absolute values of the eigenfrequencies grow at the crossover from the VLL to the LLL regime because the ratio $c_{\rm T}/c_s$ grows at the crossover.

Schweikhard et al. [42] increased the rotation speed in an attempt to reach the LLL regime. They observed linear dependence of the Tkachenko eigenfrequency on small $\omega_\perp - \Omega$ as was predicted by the theory. On the basis of good quantitative agreement with the theoretical calculation for the LLL regime by Baym [43] Schweikhard et al. concluded that they have already reached the LLL regime. However, the correct value of the shear modulus C_{66} in Eq. (56) is 10 times larger than the value $(81/80\pi^4)\rho c_s^2$ obtained by Baym [43] and used for comparison. The frequencies of the observed Tkachenko mode in fact about 4 times less than correct theoretical values for the LLL regime. It is evidence that the experiment has not yet reached the LLL limit. Since experimental values of $(\omega_\perp - \Omega)/\omega_\perp$ look small enough, apparently in order to reach the LLL limit more closely, the experiment should be done with a smaller number of atoms.

Concluding this section, let us consider restrictions on the existence of the LLL regime. First, the energy of the lowest Landau level, $\hbar\Omega$, should exceed the interaction energy per particle $\beta gn \approx gn$. This yields the inequality $n \ll \hbar\Omega/g \sim n_v\hbar^2/mg$. Second, the BEC with a regular vortex lattice exists as far as the filling factor n/n_v (the number of particles per vortex) exceeds unity (see below), i.e., the inequality $n \gg \Omega/\kappa$ is required. The two inequalities determine the interval of filling factors, where the LLL regime exists:

$$\frac{\hbar^2}{mg} \gg \frac{n}{n_v} \gg 1. \tag{59}$$

[2)] In order to receive Eq. (56) from these papers one should use the relation $gn^2 = (H_{c2} - H)^2/8\pi\kappa^2\beta$, which follows from the Ginzburg–Landau theory in the limit $\kappa = \lambda/\xi \to \infty$.

So the LLL regime is observable only for a weakly interacting Bose gas when $g \ll \hbar^2/m$. The latter inequality means that the coherence length $\xi \sim \kappa/c_s \sim \hbar/\sqrt{mgn}$ exceeds the interparticle distance $\sim 1/\sqrt{n}$.

What should happen with the LLL regime when the filling factor n/n_v approaches unity? Possible answers to this question were investigated by theoreticians both numerically and analytically [41, 43, 44]. They expect melting of the vortex lattice and destruction of the Bose condensate. Naturally the Tkachenko mode would disappear in this case.

8. Tkachenko waves in a superfluid cold atom Fermi gas. The theory of Tkachenko waves, both in an incompressible and in a compressible liquid, can be extended on Fermi superfluids [45]. The difference between a Fermi and a Bose superfluid is in the equation of state, which connects the chemical potential and the particle density. However, numerical calculations by Watanabe et al. [45] of equations similar to Eqs. (42) and (43) have shown that the difference in the equation of state has a weak effect on the Tkachenko mode eigenfrequencies. At the same time Watanabe et al. pointed out that in a Fermi superfluid gas of cold atoms it is easier to provide a larger number of particles in the cloud. This can help to reach conditions when the effect of compressibility is weaker and it is easier to determine the circulation quantum from the Tkachenko eigenfrequencies. Tkachenko waves in a superfluid Fermi gas of cold atoms are still waiting their experimental observation.

9. Conclusion. The Tkachenko wave predicted a half a century ago remains an object of intensive theoretical and experimental investigations because they provide a valuable information on properties of the ordered vortex lattices. Now these investigations focus on rotating cold atom superfluids. Here the first unambiguous observation of the Tkachenko mode was carried out. Nowadays challenges for the experiments are observation of the Tkachenko wave in the lowest Landau level regime (mean-field quantum Hall regime) and in the Fermi superfluid gases. Since the existence of the Tkachenko mode is intimately connected with the crystalline order in the vortex array, the Tkachenko mode can be a probe of the vortex lattice melting at small filling factors.

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