

Subcritical mirror structures in an anisotropic plasma

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Based on Grad–Shafranov-like equations, a gyrotropic plasma where the pressures in the static regime are only functions of the amplitude of the local magnetic field is shown to be amenable to a variational principle with a free energy density given by the parallel tension. This approach is used to demonstrate that small-amplitude static holes constructed slightly below the mirror instability threshold identify with lump solitons of KPII (Kadomtsev–Petviashvili) equation and turn out to be unstable. It is also shown that regularizing effects such as finite Larmor radius corrections cannot be ignored in the description of large-amplitude mirror structures.

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1. Introduction. Pressure-balanced structures are commonly observed in space plasmas. They are often associated with the nonlinear saturation of the mirror instability (MI) [1, 2] which, being of subcritical type [3, 4], permits the persistence of non-zero solutions below threshold [5, 6]. Furthermore, near the MI threshold, the dynamics of weakly nonlinear mirror modes are governed by an asymptotic equation of gradient type [3, 4]. This property implies an irreversible character of the mirror modes behavior, associated with ion Landau damping, where the free energy can only decrease in time. In this framework, above the threshold, the mirror modes have a blow-up behavior with a possible saturation at an amplitude level comparable to that of the ambient field. Below threshold, all stationary (localized) structures were predicted to be unstable.

The main goal of this paper is to study stationary localized structures resulting from the balance of magnetic and (both parallel and perpendicular) thermal pressures, whose simplest description is provided by anisotropic magnetohydrodynamic (MHD). Isotropic MHD equilibria are classically governed by the Grad–Shafranov (GS) equation [7–9]. We here revisit this approach in the case of anisotropic electron and ion fluids where the perpendicular and parallel pressures are given by equations of state appropriate for the static character of the solutions. However, the MHD station-

ary equations, at least in the two-dimensional geometry, turn out to be ill-posed. As a consequence, these equations require some regularization. As done in a similar context of pattern formation [10], an additional linear term involving a square Laplacian is added. For nonlinear mirror modes, regularization can originate from finite Larmor radius (FLR) corrections, which are not retained in the present analysis based on the drift kinetic equation (see, e.g. [3, 4]). The paper is organized as follows. In Section 2, the anisotropic Grad–Shafranov equations are revisited when the gyrotropic pressures depend only on the local magnetic field amplitude that, as shown in the forthcoming sections, is specific for nonlinear mirror modes. In this case, as well known [11–14], the parallel component of the equation is satisfied identically. In Section 3, we show that in the two-dimensional geometry, the problem is expressed in a variational form with a free energy given by the space integral of the parallel tension. In Section 4, it is shown that the equations of state resulting from an adiabatic approximation of the drift kinetic description, require a regularization due to an overestimate of the contributions from the particles with a large magnetic moment. We discuss in particular the small-amplitude regime and show that the pressure-balanced structures are then governed by the KPII (Kadomtsev–Petviashvili) equation which possesses lump solutions. Numerical simulations reproduce these special structures, that turn out to be unstable. Computation of stable solutions lead to large-

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amplitude purely one-dimensional solutions that appear to be sensitive to the regularization process, an indication that the regime cannot be captured by the drift kinetic approximation and that finite Larmor corrections and trapped particles are to be retained. Section 5 is the conclusion.

2. Anisotropic Grad-Shafranov equations. Gyrotropic pressure balance. We start from the pressure balance equation for a static gyrotropic MHD equilibrium

$$0 = -\nabla \cdot \mathbf{P} + \frac{1}{c} [\mathbf{j} \times \mathbf{B}], \quad (1)$$

where the current \mathbf{j} is defined from the Maxwell equation as $\mathbf{j} = \frac{c}{4\pi} \nabla \times \mathbf{B}$, and the pressure tensor \mathbf{P} is assumed to be gyrotropic. The solvability conditions read $\mathbf{B} \cdot (\nabla \cdot \mathbf{P}) = 0$, and $\mathbf{j} \cdot (\nabla \cdot \mathbf{P}) = 0$.

In terms of the tension tensor $\Pi_{ij} = \Pi_\perp(\delta_{ij} - b_i b_j) + \Pi_\parallel b_i b_j$, Eq. (1) takes the divergence form $\frac{\partial}{\partial x_j} \Pi_{ij} = 0$. Here $\mathbf{b} = \mathbf{B}/B$ is the unit vector along magnetic field and $\Pi_\perp = p_\perp + B^2/(8\pi)$ and $\Pi_\parallel = p_\parallel - B^2/(8\pi)$, where the perpendicular and parallel pressures $p_\perp = \Sigma_\alpha p_{\perp\alpha}$ and $p_\parallel = \Sigma_\alpha p_{\parallel\alpha}$ are the sum of the contributions of the various particle species α . They are expressed as $p_{\perp\alpha} = m_\alpha B^2 \int \mu f_\alpha dv_\parallel d\mu$ and $p_{\parallel\alpha} = m_\alpha B \int v_\parallel^2 f_\alpha dv_\parallel d\mu$, in terms of the distribution functions f_α , which satisfy the stationary drift kinetic equations

$$v_\parallel \nabla_\parallel f_\alpha - \left(\mu \nabla_\parallel B + \frac{e_\alpha}{m_\alpha} \nabla_\parallel \phi \right) \frac{\partial f_\alpha}{\partial v_\parallel} = 0, \quad (2)$$

where $\nabla_\parallel = \mathbf{b} \cdot \nabla$ denotes the gradient along magnetic field, v_\parallel the parallel component of the particle velocity, ϕ the electric potential, and $\mu = v_\perp^2/(2B)$ the adiabatic invariant (magnetic moment) which plays the role of a parameter. These equations are supplemented by the quasi-neutrality condition $\sum_\alpha e_\alpha B \int f_\alpha dv_\parallel d\mu = 0$, that allows one to eliminate the electric potential.

We consider partial solutions of the stationary kinetic equations (2) which are expressed in terms of two integrals of motion: the energy of the particles $W_\alpha = v_\parallel^2/2 + \mu B + (e_\alpha/m_\alpha)\phi$ and their magnetic moment μ . In general, the solution can also depend on integrals which label the magnetic field lines [12]. The choice $f_\alpha = f_\alpha(W_\alpha, \mu)$, as it will be shown in Section 4, can be matched with the solution found perturbatively for weakly nonlinear mirror modes [3, 4]. In this case the parallel and perpendicular pressures for the individual species and also the total pressures are functions of B only. We write $p_{\perp\alpha} = p_{\perp\alpha}(B)$ and $p_{\parallel\alpha} = p_{\parallel\alpha}(B)$. As seen in the next subsection, this property plays a very central role in the forthcoming analysis.

Identity in the parallel direction. The anisotropic pressure balance equation reads [15]

$$-\nabla \left(p_\perp + \frac{B^2}{8\pi} \right) + \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi} + \\ + \mathbf{B} (\mathbf{B} \cdot \nabla) \left(\frac{p_\perp - p_\parallel}{B^2} \right) = 0. \quad (3)$$

Projection along the magnetic field gives

$$-\nabla_\parallel p_\parallel - \frac{4\pi (p_\perp - p_\parallel)}{B^2} \nabla_\parallel \frac{B^2}{8\pi} = 0, \quad (4)$$

which coincides with Eq. (9.2) of Shafranov's review [9]. It is possible to prove that the solvability condition (4) reduces to an identity by means of both stationary kinetic equations (2) and the quasi-neutrality condition (see, for instance, [11–14]). Since the pressures depend on B only, Eq. (4) reduces to

$$-\frac{dp_\parallel}{dB} = \frac{p_\perp - p_\parallel}{B}. \quad (5)$$

The existence of this identity means that for stationary states, the pressure balance provides only two scalar equations which, together with the condition $\nabla \cdot \mathbf{B} = 0$, leads to a closed system of three equations for the three magnetic field components.

Defining $\nabla_\perp = \nabla - B^{-2} \mathbf{B} (\mathbf{B} \cdot \nabla)$, the perpendicular component of Eq. (3) reads

$$-\nabla_\perp p_\perp + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \nabla_\perp \frac{B^2}{8\pi} + \\ + \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] \frac{[(\nabla \times \mathbf{B}) \times \mathbf{B}]}{4\pi} = 0, \quad (6)$$

which coincides with Eq. (9.3) of Shafranov's review [9].

The two-dimensional problem. In two dimensions, we define the stream function ψ (or vector potential), such that $B_x = \partial\psi/\partial y$, $B_y = -\partial\psi/\partial x$. In terms of ψ and B_z ,

$$[(\nabla \times \mathbf{B}) \times \mathbf{B}] = \mathbf{e}_x \left(-\frac{1}{2} \frac{\partial B_z^2}{\partial x} - \frac{\partial \psi}{\partial x} \Delta \psi \right) + \\ + \mathbf{e}_y \left(-\frac{1}{2} \frac{\partial B_z^2}{\partial y} - \frac{\partial \psi}{\partial y} \Delta \psi \right) - \mathbf{e}_z \{\psi, B_z\}, \quad (7)$$

where $\{\psi, B_z\}$ denotes the Jacobian. Furthermore, $\nabla_\perp = \nabla - \frac{1}{B^2} \mathbf{B}_\perp (\mathbf{B}_\perp \cdot \nabla) - \frac{B_z}{B^2} \mathbf{e}_z (\mathbf{B}_\perp \cdot \nabla)$, where $\nabla \equiv (\partial_x, \partial_y)$ and $\mathbf{B}_\perp = (B_x, B_y)$.

In Eq. (6), we now separate the (x, y) -components:

$$-\nabla p_\perp + \frac{1}{B^2} \mathbf{B}_\perp (\mathbf{B}_\perp \cdot \nabla) p_\perp + \\ + \frac{1}{2B^2} (p_\perp - p_\parallel) \left[\nabla - \frac{1}{B^2} \mathbf{B}_\perp (\mathbf{B}_\perp \cdot \nabla) \right] B^2 + \\ + \frac{1}{4\pi} \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] \left(-\frac{1}{2} \nabla B_z^2 - \nabla \psi \Delta \psi \right) = 0. \quad (8)$$

Due to identity (5), the equation for the z component can be written

$$\begin{aligned} & \frac{B_z}{4\pi} \left[(\mathbf{B}_\perp \cdot \nabla) \left(1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right) \right] + \\ & + \frac{1}{4\pi} \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] (\mathbf{B}_\perp \cdot \nabla) B_z = 0. \end{aligned} \quad (9)$$

In terms of ψ , after integration, it leads to

$$\frac{B_z}{4\pi} \left(1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right) = f(\psi). \quad (10)$$

Interestingly, in the isotropic case ($p_\perp - p_\parallel = 0$), we have $B_z = B_z(\psi)$, in full agreement with the Grad–Shafranov reduction [7–9]. Furthermore, because the projection of the full equation on \mathbf{B} is equal to zero, in the 2D case where the fields are functions of x and y only, the projection of Eq. (8) on \mathbf{B}_\perp vanishes identically. Therefore the relevant information is obtained by taking the vector product of Eq. (8) with \mathbf{B}_\perp , in the form

$$\begin{aligned} & \left(\nabla \psi \cdot \nabla \left(p_\perp + \frac{B_z^2}{8\pi} \right) \right) - \\ & - \frac{p_\perp - p_\parallel}{2B^2} (\nabla \psi \cdot \nabla (B^2 - B_z^2)) \\ & = -\frac{B^2 - B_z^2}{4\pi} \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] \Delta \psi. \end{aligned} \quad (11)$$

This equation is supplemented by relation (10).

Equation (11) can be viewed as analogous to the Grad–Shafranov equation, the main difference being that the pressures are here prescribed as functions of the magnetic field amplitude. In particular, it does not reduce in the isotropic case to the usual Grad–Shafranov equation.

3. Variational principle. We now consider the purely two-dimensional geometry where $B_z = 0$. In this regime, $B^2 = |\mathbf{B}_\perp|^2$ and Eq. (11) reduces to

$$0 = \frac{4\pi}{B^2} \left[\nabla \psi \cdot \left(B \nabla \frac{p_\perp}{B} + \frac{p_\parallel}{B} \nabla B \right) \right] + \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] \Delta \psi. \quad (12)$$

By means of Eq. (5), it is easily checked that

$$\nabla \frac{4\pi}{B^2} (p_\perp - p_\parallel) = \frac{4\pi}{B^2} \left(B \nabla \frac{p_\perp}{B} + \frac{p_\parallel}{B} \nabla B \right).$$

Therefore, Eq. (12) takes the form

$$\nabla \cdot \left\{ \left[1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel) \right] \nabla \psi \right\} = 0 \quad (13)$$

and thus derives from the variational principle $\delta \mathcal{F} = 0$ with $\mathcal{F} = \frac{1}{4\pi} \int g(|\nabla \psi|^2) dx dy$. It rewrites $\nabla g'(B^2) \nabla \psi = 0$, where the function g is found by integrating

$$g'(B^2) = 1 + \frac{4\pi}{B^2} (p_\perp - p_\parallel).$$

Due to identity (5), we have

$$\mathcal{F} = \int \left(\frac{B^2}{8\pi} - p_\parallel \right) dx dy \equiv - \int \Pi_\parallel dx dy. \quad (14)$$

It follows that all the two-dimensional stationary states in anisotropic MHD are stationary points of the functional \mathcal{F} . Its density is a function of $B = |\nabla \psi|$ only. In the special case of cold electrons, this free energy turns out to identify with the Hamiltonian of the static problem [16].

Equations similar to (13) arise in the context of pattern structures in thermal convection. As shown in [10], such equations represent integrable hydrodynamic systems. As in the usual one-dimensional gas dynamics, these systems display breaking phenomena where the solution loses its smoothness at finite distance, due to the formation of folds. As a consequence, these models require some regularization. For patterns, the authors of [10] supplement in the equation an additional linear term involving a square Laplacian. In our case, this procedure corresponds to the replacement of \mathcal{F} by $\mathcal{F} + (\nu/2) \int (\Delta \psi)^2 dx dy$, with a constant $\nu > 0$. In plasma physics, regularization can originate from finite Larmor radius (FLR) corrections, which are not retained in the present analysis based on the drift kinetic equation (see, e.g. [3, 4]).

4. Adiabatic approximation. Equations of state and their regularization. To specify the model, we consider stationary structures which result from the nonlinear evolution of mirror modes whose characteristic frequencies are much smaller than the ion gyrofrequency. In order to characterize the (stationary) particle distribution functions governed by Eq. (2), we need to connect them to the initial state (where both ions and electrons species are assumed bi-Maxwellian). In the near-threshold weakly nonlinear regime, the transition from the initial homogeneous state is slow in time, so that, to leading order, the distribution function f_α as a function of μ and W_α retains its form during the evolution [15]. Therefore, the function $f_\alpha(\mu, W_\alpha)$ can be determined by matching with the initial distribution function $f_\alpha^{(0)} = A_\alpha \exp[-v_\parallel^2/v_{\parallel\alpha}^2 - \mu B_0 m_\alpha / T_{\perp\alpha}]$, which corresponds to $\phi = 0$ and $W_\alpha = v_\parallel^2/2 + \mu B_0$, where $A_\alpha = n_0 m_\alpha / (2\pi\sqrt{\pi} v_{\parallel\alpha} T_{\perp\alpha})$. Here $T_{\parallel\alpha}$ and $T_{\perp\alpha}$ are the initial perpendicular and transverse temperatures, B_0 – the initial homogeneous magnetic field, and $v_{\parallel\alpha} = (2T_{\parallel\alpha}/m_\alpha)^{1/2}$ – the parallel thermal velocity. As a result of the matching, we get [15]

$$f_\alpha(\mu, W_\alpha) = A_\alpha \exp \left[-\frac{2W_\alpha}{v_{\parallel\alpha}^2} + \mu B_0 m_\alpha \left(\frac{1}{T_{\parallel\alpha}} - \frac{1}{T_{\perp\alpha}} \right) \right]. \quad (15)$$

Note that this function has the Boltzmann form with respect to W_α but display, at fixed W_α , an exponential growth relatively to μ when $T_{\perp\alpha} > T_{\parallel\alpha}$, a necessary condition for MI. Such a growth, however, leads to a singular behavior of the pressures as functions of B . Indeed, for the distribution function (15), the parallel pressure is [15]:

$$p_{\parallel} = n_0(T_{\parallel i} + T_{\parallel e}) \frac{1+u}{(1+a_e u)^{c_e} (1+a_i u)^{c_i}}, \quad (16)$$

where $u = B/B_0 - 1$, $a_\alpha = T_{\perp\alpha}/T_{\parallel\alpha}$ is the parameter characterizing the anisotropy of distribution function f_α , and $c_\alpha = T_{\parallel\alpha}(T_{\parallel e} + T_{\parallel i})^{-1}$ in the case of a proton-electron plasma. The singularities at $u = -a_\alpha^{-1}$ correspond to the magnetic field

$$B_s = B_0 \frac{a_\alpha - 1}{a_\alpha} < B_0. \quad (17)$$

In the limiting case of cold electrons, $p_{\parallel} = n_0 T_{\parallel} (1+u)(1+au)^{-1}$ displays a pole singularity. Here, T_{\parallel} and the anisotropy parameter a correspond to ions only. Such an equation of state was previously derived by a quasi-normal closure of the fluid hierarchy [16].

The above singularities are presumably related to an overestimated contribution from large μ , corresponding either to small B or to large a transverse kinetic energy. In both cases, the applicability of the drift approximation breaks down and we are thus led to introduce some cut-off type correction near μ_α^* . In a simple variant, we take $f_\alpha = \tilde{C}_\alpha \exp(-m_\alpha W_\alpha/T_{\parallel\alpha})$ at $\mu > \mu_\alpha^*$, with some positive constant \tilde{C}_α , and f_α retains its original form (15) for $\mu \leq \mu_\alpha^*$. For cold electrons, the parallel ion pressure is modified into $p_{\parallel} = n_0 T_{\parallel} G(B, r)$ with

$$G(B, r) = \frac{1}{1+C} \left[\frac{(B_0 - B_s)B}{B_0(B - B_s)} R(B, r) + C e^{r(B_0 - B)} \right],$$

and

$$R(B, r) = \frac{\exp[-r(B - B_s)] - 1}{\exp[-r(B_0 - B_s)] - 1}.$$

Here, C is a (small) constant, and $r = m\mu^*/T_{\parallel}$. Noticeably, regularization leads to a non-singular positive pressure for all B , including when $B \rightarrow 0$. The modification for p_{\parallel} in the case of hot electrons is not specified here because the expressions are algebraically much more cumbersome but do not involve any additional difficulty.

KP soliton. We now show that the functional \mathcal{F} we previously introduced has the meaning of a free energy. In the weakly nonlinear regime near the MI threshold, the temporal behavior of the mirror modes can be described by a 3D model [3, 4, 15], that in the present 2D geometry reads

$$u_t = -\widehat{|k_y|} \frac{\delta F}{\delta u} \quad (18)$$

with the free energy

$$F = \int \left[\frac{1}{2} \left(-\varepsilon u^2 + u \frac{\partial_z^2}{\Delta_\perp} u + (\nabla_\perp u)^2 \right) + \frac{\lambda}{3} u^3 \right] dr. \quad (19)$$

Here u denotes the dimensionless magnetic field fluctuations and ε the distance from MI threshold. The third term in F originates from the FLR corrections, and λ is a nonlinear coupling coefficient which is positive for bi-Maxwellian distributions. In Eq. (18), the operator $\widehat{|k_y|}$ is a positive definite operator (in the Fourier representation it reduces to $|k_y|$), so that Eq. (18) has a generalized gradient form.

Let us now show that this result can be obtained from the functional \mathcal{F} defined in (14). We isolate the perturbation φ in the stream function $\psi = -B_0(x + \varphi)$ with $\varphi \rightarrow 0$ as $|\mathbf{r}| \rightarrow \infty$, so that the mean magnetic field \mathbf{B}_0 is directed along the y -axis. We then expand Eq. (14) in series with respect to u . For the sake of simplicity, we restrict the analysis to the case of cold electrons. The expansion of the integrand $B^2/(8\pi) - p_{\parallel}$ in F has then the form

$$\begin{aligned} n_0 T_{\parallel} \left[\frac{(u+1)^2}{\beta_{\parallel}} - \frac{1+u}{1+au} \right] = \\ = n_0 T_{\parallel} \left[\left(\beta_{\parallel}^{-1} - 1 \right) + u \left(a + 2\beta_{\parallel}^{-1} - 1 \right) + \right. \\ \left. + u^2 \left(-a^2 + a + \beta_{\parallel}^{-1} \right) - u^3 a^2 (a-1) + \dots \right], \end{aligned} \quad (20)$$

where we use the usual notation $\beta_{\parallel} = 8\pi n_0 T_{\parallel}/B_0^2$.

As well known (see, e.g., [3, 4]), near threshold, MI develops in quasi-transverse directions relative to \mathbf{B}_0 . This means that, in the 2D geometry, $\varphi_x \gg \varphi_y$ and, with a good accuracy, u coincides with φ_x . However, in the expansion of $u = \sqrt{(\varphi_x + 1)^2 + \varphi_y^2} - 1 \simeq \varphi_x + \varphi_y^2/2$, it is necessary to keep the second term, quadratic with respect to φ . The linear term in the expansion of \mathcal{F} vanishes and the quadratic terms is given by

$$\mathcal{F}_2 = n_0 T_{\parallel} \int \left\{ \left[a(a-1) + \frac{1}{\beta_{\parallel}} \right] \varphi_x^2 + \left[a-1 + \frac{2}{\beta_{\parallel}} \right] \varphi_y^2 \right\} dx dy,$$

where the factor $a(a-1) + 1/\beta \equiv -\varepsilon/2$ defines the MI threshold $a = 1 + 1/\beta_{\perp}$ (that the present equations of state accurately reproduces). It is also seen that for $|\varepsilon| \ll 1$, $\varphi_x/\varphi_y \sim |\varepsilon|^{-1/2}$, in agreement with the quasi-one-dimensional development of MI near threshold. In this case, \mathcal{F}_2 coincides with the quadratic term in (19), up to a simple rescaling and to the FLR contribution. Furthermore, the cubic term in (20) gives the nonlinear coupling coefficient $\lambda = a(a-1) > 0$. As a consequence, \mathcal{F} , introduced in the previous section, reduces to the free

energy of the asymptotic model. The temporal equation for φ has also the generalized gradient form originating from (18),

$$\varphi_t = -\Gamma \frac{\delta F}{\delta \varphi} \text{ with } \Gamma = -\frac{\widehat{|k_y|}}{k_x^2}, \quad (21)$$

for which the associated stationary equation reads

$$\varepsilon \varphi_{xx} + \varphi_{xxxx} - \varphi_{yy} - \lambda \partial_x (\varphi_x^2) = 0, \quad (22)$$

where the linear operator $L = -\varepsilon \partial_{xx} + \partial_{yy} - \partial_{xxxx}$ is elliptic or hyperbolic depending on the sign of ε . For $\varepsilon > 0$ (above threshold), this operator is hyperbolic, while below threshold it is elliptic and thus invertible in the class of functions vanishing at infinity. Remarkably, in the latter case, Eq. (22) identifies with the soliton for KP equation called lump. In standard notations, lump is indeed a solution of the stationary KPII equation,

$$-Vu_{xx} + u_{xxxx} - u_{yy} + 3(u^2)_{xx} = 0, \quad (23)$$

where V is the lump velocity. When comparing this equation with (22) we see that $-|\varepsilon|$ plays the role of the lump velocity V and $\lambda \varphi_x \rightarrow -3u$.

The lump solution was first discovered numerically by Petviashvili [17] using the method now known as the Petviashvili scheme (see the next section). The analytical solution was later obtained in [18]. In our notation, it reads

$$\varphi_x = -\frac{12|\varepsilon|}{\lambda} \frac{3 + \varepsilon^2 y^2 - |\varepsilon| x^2}{[3 + \varepsilon^2 y^2 + |\varepsilon| x^2]^2}.$$

This function vanishes algebraically at infinity like r^{-2} . In the center region $-|\varepsilon|^{-2} \sqrt{|\varepsilon| x^2 - 3} < y < |\varepsilon|^{-2} \sqrt{|\varepsilon| x^2 - 3}$, the magnetic field displays a hole with a minimum at $x = y = 0$ equal to $-4|\varepsilon|/\lambda$. In the outer region, the magnetic lump has two symmetric humps with maximum values $|\varepsilon|/(2\lambda)$ at $y = 0$ and $x = \pm 3|\varepsilon|^{-1/2}$. The main contribution to the “skewness” $I = \int \varphi_x^3 dx dy$ comes from the hole region, providing a negative value to I , in complete agreement with [3, 4].

Numerical solutions – the methods. In the 2D case, our regularized model equation for stationary pressure-balanced structures has a variational form

$$-\partial_x \left(\frac{1 + \varphi_x}{1 + u} \frac{dg}{du} \right) - \partial_y \left(\frac{\varphi_y}{1 + u} \frac{dg}{du} \right) + \nu \Delta^2 \varphi = 0. \quad (24)$$

Clearly, Eq. (24) describes stationary points $\delta \mathcal{F}/\delta \varphi = 0$ of the functional $\mathcal{F} = \int [g(u) + (\nu/2)(\Delta \varphi)^2] dx dy$, with some constant parameter ν (in this expression and everywhere below, we use dimensionless variables).

We applied two numerical methods to solve Eq. (24). The first one is a generalization of the well known gradient method which corresponds to a dissipative dynamics along an auxiliary time-like variable τ of the form $\varphi_\tau = -\widehat{\Gamma}(\delta \mathcal{F}/\delta \varphi)$, with a positive definite linear operator $\widehat{\Gamma}$. It is clear that attractors in the phase space of the above dynamical system are stable solutions of Eq. (24). Unstable solutions however cannot be found by this method.

Furthermore, the linear part of Eq. (24) is of the form $\widehat{L}\varphi = -g''(0)\varphi_{xx} - g'(0)\varphi_{yy} + \nu \Delta^2 \varphi$. The coefficient $g''(0)$ is proportional to ε (introduced in the previous section) and $g'(0)$ is positive within the adiabatic approximation. When these two are positive, the operator L is elliptic and it is possible to employ the so-called Petviashvili method [17]. It is a specific method for finding localized solutions of equations of the form $\widehat{M}\varphi = N[\varphi]$, with a positively definite linear operator \widehat{M} and a nonlinear part $N[\varphi]$. Note that in our case the Fourier image of \widehat{M} is

$$M(k_x, k_y) = g''(0)k_x^2 + g'(0)k_y^2 + \nu(k_x^2 + k_y^2)^2 > 0. \quad (25)$$

In its simplest form, the iteration scheme of the Petviashvili method reads

$$\varphi_{n+1} = (\widehat{M}^{-1} N[\varphi_n]) \left(\frac{\int \varphi_n \widehat{M} \varphi_n dx dy}{\int \varphi_n N[\varphi_n] dx dy} \right)^{-\gamma}, \quad (26)$$

where γ is a positive parameter in the range $1 < \gamma < 2$. The corresponding multiplier strongly affects the structure of attractive regions in the phase space.

It is worth noting that if the operator \widehat{L} is hyperbolic, solutions of the problem are not localized with respect to both x and y coordinates, and will be periodic or more generally quasiperiodic [19, 20].

The results. We performed computations with both numerical methods using fast Fourier transform numerical routines for the evaluation of the linear operators. Periodic boundary conditions for a computational square $2\pi \times 2\pi$ were assumed.

For the gradient method, we used the simplest first-order Euler scheme for stepping along τ , with $\delta\tau \sim 0.01$. The operator $\widehat{\Gamma}$ was taken in a form giving stable computation, namely $\Gamma(k_x, k_y) = 1/[k_x^2 + k_y^2 + \nu(k_x^2 + k_y^2)^2]$.

As for the Petviashvili method, the value $\gamma = 1.8$ was used, leading, after an erratic transient, to a convergence of the iterations to unstable solutions of the variational equation (24).

The main results of our computations can be formulated as follows. There do exist unstable localized solutions of Eq. (24), which are similar to the lump solutions of KPII equation, when written in terms of $u = \partial_x \varphi$

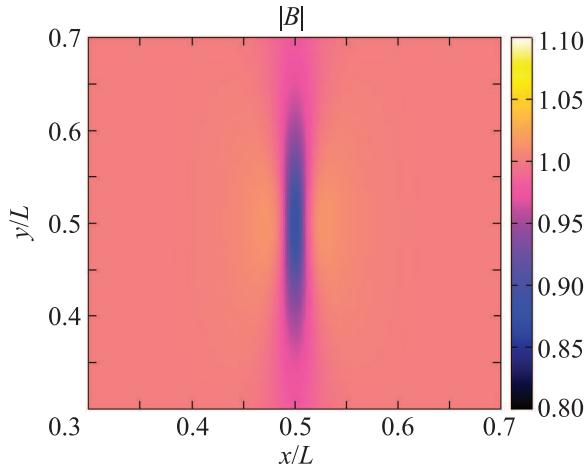


Fig. 1. Unstable localized solution for $\nu = 0.0004$, $r = 7$, $B_s = 0.5$ (in units of B_0), and $C = 0.002$. The value $1/\beta_{\parallel} = 1.127$ prescribes an aspect ratio $\sqrt{g''(0)/g'(0)} = 0.2$

(Fig. 1). For asymptotically small ε , they accurately coincide with KP solutions, independently of the electron temperature, as it should be. Such low-amplitude stationary states do not depend on the particular choice of the regularization of $g(u)$. No other kinds of solutions were found with the Petviashvili method.

When the gradient method is used, large amplitudes $u \sim 1$ are achieved in many cases, and the final result turns out to be dependent on the choice of the parameters r and C in the regularized function g . Without regularization, no smooth stationary state is approached. Instead, a singularity occurs. Differently, when a regularized g with parameters $r \sim 10$ and $C \sim 0.001$ is used, the final state identifies with a one-dimensional stripe in the form of a magnetic hole, as shown in Fig. 2 that also displays typical stages of the “gradient” evolution. In all simulations, the magnetic field in the stripe was smaller than the “singular” magnetic field B_s given by Eq. (17). For increasing r , the magnetic field in the stripe tends to decrease, down to 0. For initial conditions in the form of a slightly perturbed 2D lump, the final result is always a one-dimensional stripe of hole type, which demonstrates the instability of the 2D lump, in full agreement with the analytical prediction [3, 4].

In no cases stable 2D structures localized both in x and y directions were found. Instead, the gradient method showed that stable structures can only be one-dimensional, transverse to the magnetic field. An initial localized perturbation of sufficiently high amplitude develops into an increasingly long structure along the y axis, and eventually reaches the boundary of the computational domain.

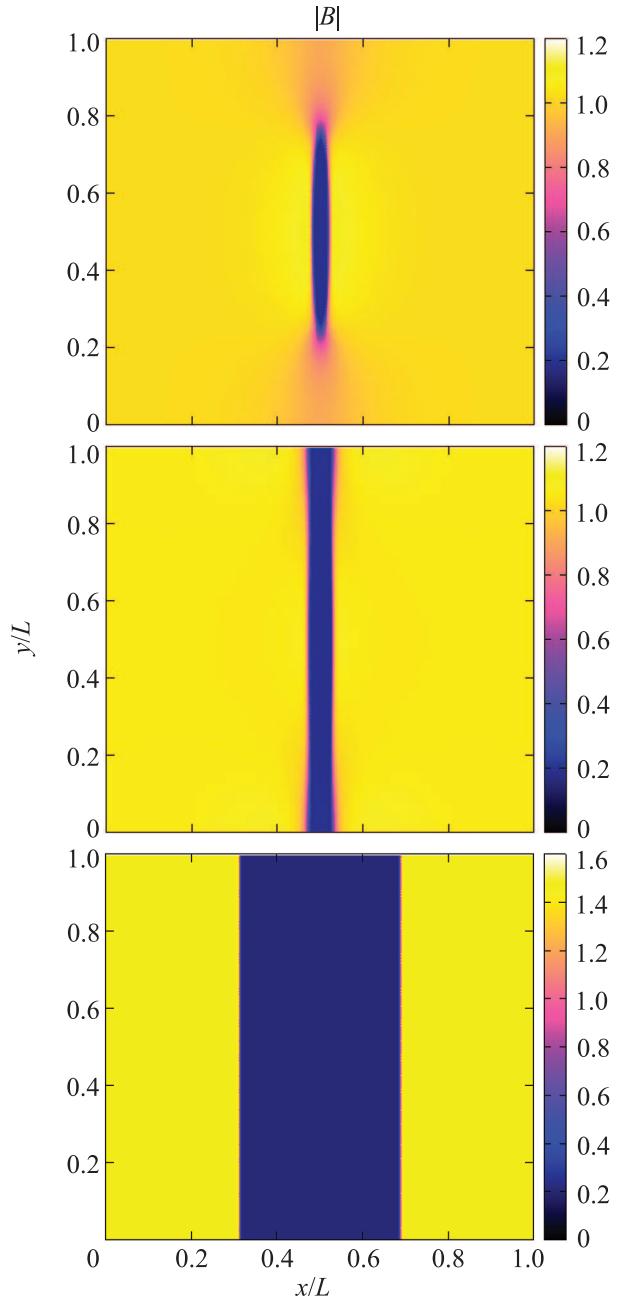


Fig. 2. Formation of a stable 1D solution in a gradient computation, for the same parameters as in Fig. 1

The question arises whether the 1D shock solutions obtained in [16] (for which $\min B > B_s$) would identify with the present solution when $\nu \rightarrow 0$, a limit which is unreachable in the present numerics. It is possible that the presence of the bi-Laplacian regularization leads to overshooting in the shock solution, resulting in the convergence towards solutions where $\min B < B_s$.

5. Conclusion. A detailed analysis was presented for the Grad–Shafranov equations describing static force-balanced mirror structures with anisotropic pres-

sures given by equations of state derived from drift kinetic equations, when assuming an adiabatic evolution from bi-Maxwellian initial conditions. It turns out that in two dimensions, the problem is amenable to a variational formulation with a free energy provided by the space integral of the parallel tension (it is worth noting that the variational formalism with the same free energy density extends to cylindrical geometry, a question that is beyond the scope of this letter and will be addressed in forthcoming works). Slightly below the mirror instability threshold, small amplitude solutions associated to KPII lumps are obtained and shown to be unstable. Differently, when considering stable subcritical structures, the drift kinetic approximation breaks down, as the deep magnetic holes obtained by a gradient method appear to be strongly sensitive to the regularization process, an effect which in a more realistic description could be provided by FLR corrections and/or particle trapping.

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1. A. A. Vedenov and R. Z. Sagdeev, *Plas. Phys. & Problem of Cont. Therm. React.*, V. III, ed. by M. A. Leontovich, Pergamon Press, NY (1958), p. 332.
2. S. P. Gary, *Theory of Space Plasma Microinstabilities*, Cambridge Atmospheric and Space Science Series (1993).
3. E. A. Kuznetsov, T. Passot, and P. L. Sulem, Phys. Rev. Lett. **98**, 235003 (2007) [JETP Lett. **86**, 637 (2007)].
4. E. A. Kuznetsov, T. Passot, and P. L. Sulem, Pis'ma v ZhETF **86**, 725 (2007).
5. J. Soucek, E. Lucek, and I. Danbouras, J. Geophys. Res. **113**, A04203 (2007).
6. V. Génot, E. Budnik, P. Hellinger, T. Passot, G. Belmont, P. Trávníček, P. L. Sulem, E. Lucek, and I. Danbouras, Ann. Geophys. **27**, 601 (2009).
7. H. Grad, *Notes on Magneto-Hydrodynamics I-III: General Fluid Equations*, CIMS, New York University, Doc. NYO-6486-I(III) (1956); H. Grad and H. Rubin, *Proceedings of the Second United Nations International Conference on the Peaceful Uses of Atomic Energy (UN, Geneva)*, V. 31, Session A-5, P/386 (1958), p. 190.
8. V. D. Shafranov, ZhETF **33**, 710 (1957) [Sov. Phys. JETP **6**, 545 (1958)].
9. V. D. Shafranov, *Reviews of Plasma Physics*, V. 2, ed. by M.A. Leontovich, Atomizdat, M. (1963), p. 92 [Consultant Bureau, N.Y. (1966), p. 103].
10. N. M. Ercolani, R. Indik, A. C. Newell, and T. Passot, Nonlinear Sci. **10**, 223 (2000).
11. T. G. Northrop and K. J. Whiteman, Phys. Rev. Lett. **12**, 639 (1964).
12. H. Grad, Phys. Fluids **9**, 498 (1966); Phys. Fluids **10**, 137 (1967).
13. L. S. Hall and B. McNamara, Phys. Fluids **18**, 552 (1975).
14. L. E. Zakharov and V. D. Shafranov, in *Reviews of Plasma Physics*, V. 11, ed. by B. B. Kadomtsev, Atomizdat, M. (1985), p. 118 [Consultant Bureau, N.Y., p. 153].
15. E. A. Kuznetsov, T. Passot, and P. L. Sulem, Pis'ma v ZhETF **96**, 716 (2012) [JETP Lett. **96**, 642 (2013)].
16. T. Passot, V. Ruban, and P. L. Sulem, Phys. Plasmas **13**, 102310 (2006).
17. V. I. Petviashvili, Fiz. Plazmy **2**, 469 (1976) [Sov. J. Plasma Phys. **2**, 247 (1976)].
18. S. V. Manakov, V. E. Zakharov, L. A. Bordag, A. B. Its, and V. B. Matveev, Phys. Lett. A **63**, 205 (1979).
19. V.E. Zakharov and E.A. Kuznetsov, ZhETF **113**, 1892 (1998) [JETP **86**, 1035 (1998)].
20. E. A. Kuznetsov and F. Dias, Phys. Rep. **507**, 43 (2011).