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Line bundle twisted chiral de Rham complex, chiral Riemann–Roch formula and D-branes on toric manifolds

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I represent the results of the elliptic genus calculations in various examples of twisted chiral de Rham complex on one- and two-dimensional toric compact manifolds. The explicit calculations are made for line bundle twisted chiral de Rham complex on $\mathbb{P}^1$, $\mathbb{P}^2$ and Hirzebruch surface. Based on these results I propose the elliptic genus expression of the bundle twisted chiral de Rham complex for general smooth compact two dimensional toric manifold. The expression resembles Riemann–Roch formula and coincides with the later in certain limit. I interprete the result in terms of infinite tower of open string oscillator contributions and identify directly the open string boundary conditions of the corresponding bound state of D-branes.

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1. Introduction. It has been proposed by J. Harvey and G. Moore [1] that sheaves can be used to model D-branes on large-radius Calabi–Yau (CY) manifolds. Since then, a significant progress has been made in understanding of sheaves as models of topological D-branes. As a review of the results and a source of necessary references see [2, 3].

In the important work of Malikov, Schechtman, and Vaintrob [4] a sheaf of vertex algebras, which is called chiral de Rham complex has been introduced for every smooth variety. When the variety is $\mathbb{C}^d$ this sheaf is known as "$bc\beta\gamma$"-system. Soon after the significant application of chiral de Rham complex in the String Theory has been represented in the beautiful paper of Borisov [5] where the chiral de Rham complex construction has been given for each pair of dual reflexive polytopes defining toric CY manifold.

In the paper [6] the generalization of Borisov construction [5] has been represented to include chiral de Rham complex on toric manifold twisted by line bundle. It was conjectured there that the cohomology of line bundle twisted chiral de Rham complex may describe an infinite tower of states in the open string sector of certain D-brane bound state on toric manifold. In that sense, the conjecture from [6] is an extended version of the suggestion of J.A. Harvey and G. Moore [1]. In defense of the conjecture the results of the paper [7] talk also, where the open string sector of Gepner model boundary states was investigated by "$bc\beta\gamma$" representations of $N=2$ minimal models [8].

In this note I represent some additional evidences in support of the conjecture from [6] and [1] calculating elliptic genus of the chiral de Rham complex twisted by line bundle on compact smooth toric two-dimensional manifold.

2. Line bundle twisted chiral de Rham complex elliptic genus. In this section a brief review of line bundle twisted chiral de Rham complex construction and elliptic genus calculation is represented for a smooth complete toric variety. For more details the reader is referred to [4–6, 9].

2.1. The elliptic genus of chiral de Rham complex. I describe first the chiral de Rham complex and elliptic genus for the complete smooth toric manifold following closely to [5, 9].

Let X be a smooth variety of dimension d. In local coordinates $x_1, ..., x_d$ the set of local sections of chiral de Rham complex $\mathrm{MSV}(X)$ on X, can be described as follows. To the coordinates $x_1, ..., x_d$ we associate "$bc\beta\gamma$" system of fields

$$a_\mu(z) = \sum_n a_\mu[n] z^{-n}, \quad a^*_\mu(z) = \sum_n a^*_\mu[n] z^{-n-1},$$
$$\alpha_\mu(z) = \sum_n \alpha_\mu[n] z^{-n-1/2}, \quad \alpha^*_\mu(z) = \sum_n \alpha_\mu[n] z^{-n-1/2}, \tag{1}$$

where $\mu = 1, ..., d$ with the following nontrivial supercommutators between the modes

$$[a^*_\mu[n], a_\nu[m]]_- = \delta_{\mu,\nu}\delta(n+m),$$
$$[\alpha^*_\mu[n], \alpha_\nu[m]]_+ = \delta_{\mu,\nu}\delta(n+m). \tag{2}$$

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Then the set of local sections of the chiral de Rham complex is generated by the creation operators of the fields (1) from the vacuum state $|0\rangle$ which is defined by

$$a_\mu[n]|0\rangle = a^*_\mu[n-1]|0\rangle = \alpha_\mu[n]|0\rangle = \\ = \alpha^*_\mu[n-1]|0\rangle = 0, \quad n>0. \tag{3}$$

The important property is the behavior of the $bc\beta\gamma$-system under the local change of coordinates [4]. For each new set of coordinates

$$y_\mu = g_\mu(x_1,...,x_d),\ x_\mu = f_\mu(y_1,...,y_d) \tag{4}$$

the isomorphic $bc\beta\gamma$-system of fields is given by

$$\begin{aligned} b_\mu(z) &= g_\mu(a_1(z),...,a_d(z)),\\ \beta_\mu(z) &= \frac{\partial g_\mu}{\partial a_\nu}(a_1(z),...,a_d(z))\alpha_\nu(z),\\ \beta^*_\mu(z) &= \frac{\partial f_\nu}{\partial b_\mu}(a_1(z),...,a_d(z))\alpha^*_\nu(z),\\ b^*_\mu(z) &= \frac{\partial f_\nu}{\partial b_\mu}(a_1(z),...,a_d(z))a^*_\nu(z) +\\ &+ \frac{\partial^2 f_\lambda}{\partial b_\mu \partial b_\nu}\frac{\partial g_\nu}{\partial a_\rho}(a_1(z),...,a_d(z))\alpha^*_\lambda(z)\alpha_\rho(z). \end{aligned} \tag{5}$$

On the space of local sections of MSV(X) the $N=2$ Virasoro superalgebra is acting by

$$G^- = \sum_\mu \alpha_\mu a^*_\mu,\ G^+ = -\sum_\mu \alpha^*_\mu \partial a_\mu,\ J = \sum_\mu \alpha^*_\mu \alpha_\mu,\\ T = \sum_\mu \left[a^*_\mu \partial a_\mu + \frac{1}{2}(\partial\alpha^*_\mu \alpha_\mu - \alpha^*_\mu \partial \alpha_\mu)\right]. \tag{6}$$

Though this algebra is globally defined only for Calabi–Yau manifold [4, 5] the zero mode $L[0]$ of the field $T(z)$ and zero mode $J[0]$ of the field $J(z)$ are invariant under the local change of coordinates and hence, globally defined in general case. It provides the space of local sections of the chiral de Rham complex with the double grading.

Since the cohomology $H^*(\mathrm{MSV}(X))$ of chiral de Rham complex is finite-dimensional vector spaces at every eigenvalue of $L[0]$ the Euler characteristics of the chiral de Rham complex are well-defined. It allows to give the following definition of the elliptic genus of the chiral de Rham complex [9]

$$\mathrm{Ell}(X,y,q) = y^{-d/2}\mathrm{SuperTr}_{H^*(\mathrm{MSV}(X))}(y^{J[0]}q^{L[0]}). \tag{7}$$

When X is a complete toric variety [10, 11] the cohomology $H^*(\mathrm{MSV}(X))$ could be calculated as Čhech cohomology for the open affine covering defined by d-dimensional cones [9]. So one has to describe the toric data defining X as well as its covering (see [10, 11]).

We have a lattice Λ of rank d, its dual lattice Λ^* and a complete polyhedral fan $\Sigma \subset \Lambda$ which is a union of finite number of d-dimensional cones

$$\Sigma = \cup_I C_I \tag{8}$$

so that each intersection of the cones C_I is also a cone from Σ. The variety X is smooth if the cones C_I are simplicial and are generated by a basis in Λ. The cones C_I define the open affine covering of X

$$X = \cup_I A_I,\ A_I = \mathrm{Spec}(\mathbb{C}[C^*_I]), \tag{9}$$

where $C^*_I \subset \Lambda^*$ is dual cone to C_I [10, 11]. Intersection of any number of A_I is another open subset of this type so the covering is acyclic for MSV(X) [5]. In addition the natural action of $(\mathbb{C}^*)^d$ can be extended to MSV(X). For any affine subset A_I this action endows the sections of MSV(X) over A_I with natural grading by the lattice Λ^*. The same is true for the sections over an intersection of finite number of A_I's. Thus, we come to the expression from [9]

$$\mathrm{Ell}(X,t,y,q) = y^{-d/2}\sum_{p^*\in\Lambda^*}\sum_{I_1,...,I_k}(-1)^k t^{p^*} \times \\ \times \mathrm{sdim}_{p^*} H^0(A_{I_1}\cap ... \cap A_{I_k}, \mathrm{MSV}(X)), \tag{10}$$

where sdim is a super-dimension which is finite for each fixed $p^* \in \Lambda^*$ and powers of y and q and t is a multi-variable due to the natural action of $(\mathbb{C}^*)^d$.

One can show [9] that elliptic genus of toric manifold can be written as follows

$$\mathrm{Ell}(X,t,y,q) = y^{-d/2}\sum_{p^*\in\Lambda^*}\sum_{C\subset\Sigma}(-1)^{\mathrm{codim}C} \times \\ \times \prod_{\mu=1,...,\dim C}\frac{t^{p^*}}{1-yq^{p^*(e_\mu)}}G(y,q)^d. \tag{11}$$

2.2. The elliptic genus of line bundle twisted chiral de Rham complex. The line bundle on a toric variety is given by toric divisor support function ω^* [10, 11]. It is a piece-wise linear function on maximal dimension cones which is consistent on the intersections of the cones. In other words the function ω^* is a collection of elements $\{\omega^*_I\}$ from the lattice Λ^* which are compatible with the restriction map for every intersection $C_I \cap C_J$:

$$\omega^*_I|_{C_J} = \omega^*_J|_{C_I} \equiv \omega^*_{IJ}. \tag{12}$$

The generalization of (11) for the line bundle twisted chiral de Rham complex is very simple and given by [6]

$$\mathrm{Ell}_{\omega^*}(X,t,y,q) = y^{-d/2} \sum_{p^*\in\Lambda^*}\sum_{C\subset\Sigma}(-1)^{\mathrm{codim}C} \times$$
$$\times \prod_{\mu=1,\dots,\mathrm{dim}C} \frac{t^{p^*-\omega^*_C}}{1-yq^{p^*(e_\mu)}} G(y,q)^d, \qquad (13)$$

where ω^*_C is the restriction of ω^* on the cone C.

To explain this formula we consider first the sections of line bundle twisted chiral de Rham complex over the A_I. In this case the vacuum state is

$$|\Omega_I\rangle = \prod a_{I\mu}[0]^{-\omega^*_I(e_\mu)}|0\rangle. \qquad (14)$$

To generate the sections one has to apply the creation operators of the "$bc\beta\gamma$" (1) to vacuum $|\Omega_I\rangle$ where instead of the fields $a^*_\mu(z)$ one has to take covariant derivatives fields

$$\nabla_{I\mu}(z) = a^*_{I\mu}(z) + \omega^*_I(e_\mu) z^{-1} a^{-1}_{I\mu}(z). \qquad (15)$$

The last term in this expression is caused by a gauge potential defined on A_I. Let us denote the module generated by this way as $\mathbb{M}_I$. One can show that the vacuum $|\Omega_I\rangle \in \mathbb{M}_I$ defines trivializing isomorphism of modules (over the chiral de Rham complex on A_{C_I}) [6]

$$g_I : \mathbb{M}_I \to M_I. \qquad (16)$$

We endowed the "$bc\beta\gamma$" fields in the formulas (14), (15) by additional index I because they differ for different cones C_I.

As a consequence, g_I defines the isomorphism between a pair of $N=2$ Virasoro superalgebras, where the second one acts on M_I by the currents (6) while the first one acts on $\mathbb{M}_I$ by the currents (6) where the fields $\nabla_\mu(z)$ are taken instead of $a^*_\mu(z)$. Now the expression (13) follows from the corresponding Čhech complex of the covering [6].

One can use the isomorphisms (16) also to argue that the open string states on toric manifold with holomorphic Chan–Paton vector bundle can be described as vector bundle twisted chiral de Rham complex.

3. Line bundle twisted chiral de Rham complex elliptic genera calculations. *3.1. Elliptic genus of line bundle twisted chiral de Rham complex on $\mathbb{P}^1$.* Let e be the standard basis in $\mathbb{R}^1$. The fan Σ of $\mathbb{P}^1$ is the collection of 1-dimensional cones $C_+ \in \Lambda$, $C_- \in \Lambda$ and 0-dimensional cone $C_\bullet = C_+ \cap C_-$. The 1−dim cones generating the fun Σ are spanned by the vectors

$$C_+ = \mathrm{Cone}(e),\ C_- = \mathrm{Cone}(-e). \qquad (17)$$

The toric divisor support function ω^* defining the line bundle $O(N)$ on $\mathbb{P}^1$ is determined by its values on the generators of cones

$$\omega^*(e) = N_+,\ \omega^*(-e) = N_-, \qquad (18)$$

where $N_\pm \in \mathbb{Z}$ and $N = N_+ + N_-$.

According to (13) we find

$$\mathrm{Ell}_{\omega^*}(\mathbb{P}^1,t,y,q) = y^{-1/2} \times$$
$$\times \sum_{n\in\mathbb{Z}} \left[\frac{t^{n-N_+}}{1-yq^n}G(y,q) + \frac{t^{-n-N_-}}{1-yq^n}G(y,q) - G(y,q)\right]. \qquad (19)$$

One can rewrite this expression in terms of theta functions [9, 6]

$$\mathrm{Ell}_{\omega^*}(\mathbb{P}^1,t,y,q) =$$
$$= t^{-N_+}\frac{\Theta(ty^{-1},q)}{\Theta(t,q)} + t^{N-N_+}\frac{\Theta(t^{-1}y^{-1},q)}{\Theta(t^{-1},q)}, \qquad (20)$$

where

$$\Theta(u,q) =$$
$$= q^{1/8}\prod_{n=0}(1-u^{-1}q^{n+1})(1-uq^n)(1-q^{n+1}) =$$
$$= q^{1/8}\sum_{n\in Z}(-1)^n q^{(n^2-n)/2}u^{-n}. \qquad (21)$$

By the l'Hopital rule we find

$$\mathrm{Ell}_N(\mathbb{P}^1,y,q) \equiv \lim_{t\to 1}\mathrm{Ell}_{\omega^*}(\mathbb{P}^1,t,y,q) =$$
$$= Ny\eta(q)^{-3}\Theta(y,q) + \mathrm{Ell}(\mathbb{P}^1,y,q), \qquad (22)$$

where

$$\mathrm{Ell}(\mathbb{P}^1,y,q) = y\eta(q)^{-3}\left[\Theta(y,q) + 2y\frac{\partial\Theta(y,q)}{\partial y}\right],$$
$$\eta(q) = q^{1/24}\prod_{n=1}(1-q^n). \qquad (23)$$

One can give the following interpretation of the expression (22). The last term is just the elliptic genus of $\mathbb{P}^1$ coming from $D2$-brane wrapping $\mathbb{P}^1$. The first term gives the contribution due to nontrivial line bundle $O(N)$ is defined on $\mathbb{P}^1$. It coincides with the open string oscillators contribution coming from N toric invariant $D0$-branes on $\mathbb{P}^1$. Thus, we interprate the cohomology of chiral de Rham complex twisted by $O(N)$-bundle as open string states of the bound state of N $D0$-branes and one $D2$-brane on $\mathbb{P}^1$.

3.2. Elliptic genus of line bundle twisted chiral de Rham complex on $\mathbb{P}^2$. The fan of $\mathbb{P}^2$ is given as follows.

Let e_1, e_2 be the standard basis in $\mathbb{R}^2$. The 2-dim cones constituting the fun Σ are spanned by the vectors

$$C_{01} = \mathrm{Cone}(e_0 = -e_1 - e_2, e_1),$$
$$C_{02} = \mathrm{Cone}(e_0 = -e_1 - e_2, e_2), \qquad (24)$$
$$C_{12} = \mathrm{Cone}(e_1, e_2).$$

The toric divisor support function ω^* of the line bundle $O(N)$ is determined by its values on the vectors generating 1-dimensional cones $C_i = \mathrm{Cone}(e_i)$, $i = 0, 1, 2$,

$$\omega^*(e_1) = N_1,\ \omega^*(e_2) = N_2,\ \omega^*(e_0) = N_0, \\ N = N_0 + N_1 + N_2. \tag{25}$$

According to (13) we obtain

$$\begin{aligned} &\mathrm{Ell}_N(\mathbb{P}^2, t_1, t_2, y^{-1}, q) = \\ &= t_1^{-N_1} t_2^{-N_2} \frac{\Theta(t_1 y^{-1}, q)\Theta(t_2 y^{-1}, q)}{\Theta(t_1, q)\Theta(t_2, q)} + \\ &+ t_1^{-N_1} t_2^{N_0+N_1} \frac{\Theta(t_2^{-1} y^{-1}, q)\Theta(t_1 t_2^{-1} y^{-1}, q)}{\Theta(t_2^{-1}, q)\Theta(t_1 t_2^{-1}, q)} + \\ &+ t_1^{N_0+N_2} t_2^{-N_2} \frac{\Theta(t_1^{-1} y^{-1}, q)\Theta(t_1^{-1} t_2 y^{-1}, q)}{\Theta(t_1^{-1}, q)\Theta(t_1^{-1} t_2, q)}. \end{aligned} \tag{26}$$

By the l'Hopital rule we find

$$\begin{aligned} &\mathrm{Ell}_N(\mathbb{P}^2, y, q) \equiv \lim_{t_1, t_2 \to 1} \mathrm{Ell}_{\omega^*}(\mathbb{P}^2, t_1, t_2, y, q) = \\ &= \frac{N^2}{2}[y\eta(q)^{-3}\Theta(y, q)]^2 + \\ &+ \frac{3N}{2}[y\eta(q)^{-3}\Theta(y, q)]\mathrm{Ell}(\mathbb{P}^1, y, q) + \mathrm{Ell}(\mathbb{P}^2, y, q), \end{aligned} \tag{27}$$

where

$$\begin{aligned} &\mathrm{Ell}(\mathbb{P}^2, y, q) = \\ &= \left[\frac{9}{8} - 3q\eta(q)^{-1}\frac{\partial \eta(q)}{\partial q}\right][y\eta(q)^{-3}\Theta(y, q)]^2 + \\ &+ y^3\eta(q)^{-6}\left\{6\Theta(y, q)\frac{\partial \Theta(y, q)}{\partial y} + \right. \\ &\left. + 3y\left[\frac{\partial \Theta(y, q)}{\partial y}\right]^2 + \frac{3}{2}y\Theta(y, q)\frac{\partial^2 \Theta(y, q)}{\partial y^2}\right\} \end{aligned} \tag{28}$$

is the elliptic genus of $\mathbb{P}^2$.

We give the following D-brane interpretation of the expression above. The first contribution comes from the open string states of N^2 $D0$-branes. The second one comes from the open string states of $D2$-brane which is a divisor linearly equivalent to N multiple of the hyperplane $\mathbb{P}^1 \subset \mathbb{P}^2$, so the factor $y\eta(q)^{-3}\Theta(y, q)$ gives the open string oscillator contributions in the transverse direction to the $D2$-brane. The last term comes from $D4$-brane wrapping $\mathbb{P}^2$. Thus the cohomology of chiral de Rham complex twisted by $O(N)$ bundle describes the open string states of the bound state of $D0$–$D2$–$D4$–branes.

3.3. Elliptic genus of chiral de Rham complex on Hirzebruch surface $\mathbb{F}_k$ twisted by line-bundle. The 1-dim cones, generating the fan of the Hirzebruch surface $\mathbb{F}_k$ spanned on the vectors

$$s_1 = e_1,\ s_2 = e_2,\ s_3 = -e_1 + ke_2,\ s_4 = -e_2, \tag{29}$$

where $e_{1,2}$ is a standard orthogonal basis $\mathbb{R}^2$ and k is a positive integer number. The 2-dim cones are spanned by the pairs of vectors

$$C_{i,i+1} = \mathrm{Cone}(s_i, s_{i+1}),\ i = 1, ..., 3, \\ C_{4,1} = \mathrm{Cone}(s_4, s_1). \tag{30}$$

Toric divisor support function ω^* assotiated to some divisor J is determined by

$$\omega^*(s_i) = N_i,\ i = 1, ..., 4. \tag{31}$$

According to (13) we obtain

$$\begin{aligned} &\mathrm{Ell}_{O(J)}(\mathbb{F}_k, t_1, t_2, y^{-1}, q) = \\ &= t_1^{N_1} t_2^{N_2} \frac{\Theta(t_1^{-1} y, q)\Theta(t_2^{-1} y, q)}{\Theta(t_1^{-1}, q)\Theta(t_2^{-1}, q)} + \\ &+ (t_1^k t_2)^{N_2} t_1^{-N_3} \frac{\Theta(t_1^{-k} t_2^{-1} y, q)\Theta(t_1 y, q)}{\Theta(t_1^{-k} t_2^{-1}, q)\Theta(t_1, q)} + \\ &+ t_1^{-N_3}(t_1^k t_2)^{-N_4} \frac{\Theta(t_1^{-1} y^{-1}, q)\Theta[(t_1^k t_2)^{-1} y^{-1}, q]}{\Theta(t_1^{-1}, q)\Theta(t_1^{-k} t_2^{-1}, q)} + \\ &+ t_1^{N_1} t_2^{-N_4} \frac{\Theta(t_2^{-1} y^{-1}, q)\Theta(t_1 y^{-1}, q)}{\Theta(t_2^{-1}, q)\Theta(t_1, q)}. \end{aligned} \tag{32}$$

Taking the limit $t_1, t_2 \to 1$ we obtain

$$\begin{aligned} &\mathrm{Ell}_{O(J)}(\mathbb{F}_k, y, q) = [1 - N_1 - N_2 - N_3 - N_4 + \\ &+ N_1 N_2 + N_1 N_4 + N_2 N_3 + N_3 N_4 + \\ &+ \frac{k}{2}(-N_4 + N_2)(1 - N_4 - N_2)]y^2[\eta(q)^{-3}\Theta(y, q)]^2 + \\ &+ [4 - 2N_1 + (k-2)N_2 - 2N_3 - \\ &- (2+k)N_4]y^3[\eta(q)^{-3}\Theta(y, q)]\left[\eta(q)^{-3}\frac{\partial \Theta(y, q)}{\partial y}\right] + \\ &+ 4y^4[\eta(q)^{-3}\frac{\partial \Theta(y, q)}{\partial y}]^2. \end{aligned} \tag{33}$$

One can see that (33) depends only on $p = -N_1 - N_3 - kN_4$ and $r = -N_2 - N_4$ and can be rewritten as

$$\begin{aligned} &\mathrm{Ell}_{O(J)}(\mathbb{F}_k, y, q) = \frac{2pr - kr^2}{2}[y\eta(q)^{-3}\Theta(y, q)]^2 + \\ &+ \frac{2p + (2-k)r}{2}[y\eta(q)^{-3}\Theta(y, q)]\mathrm{Ell}(\mathbb{P}^1, y, q) + \\ &+ \mathrm{Ell}(\mathbb{F}_k, y, q), \end{aligned} \tag{34}$$

where

$$\mathrm{Ell}(\mathbb{F}_k, y, q) = [y\eta(q)^{-3}\Theta(y, q) + 2y^2\eta(q)^{-3}\frac{\partial \Theta(y, q)}{\partial y}]^2. \tag{35}$$

Recall that the Chow group [10, 11] $A_1(\mathbb{F}_k)$ is given by the factor of the lattice $\mathbb{Z}^4 = \{(N_1, ..., N_4)\}$ by the sublattice generated by the linear functions $m_1 e_1^* + m_2 e_2^*$:

$$A_1(\mathbb{F}_k) = \{(N_1, ..., N_4)/(N_3 = -N_1 + kN_2,\ N_4 = -N_2)\}. \tag{36}$$

Hence one can take the divisors

$$J_1 = s_1, \; J_2 = s_2 \tag{37}$$

with intersection matrix

$$C_{ij} = J_i \cdot J_j = \begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix} \tag{38}$$

as the generators of the Chow group classes. The correspondence between one dimensional cones of the fan and Weyl divisors on toric manifold [10, 11] is implied in (37). Hence the expression (35) takes the following form

$$\begin{aligned} \mathrm{Ell}_{O(J)}(\mathbb{F}_k, y, q) = \frac{J^2}{2}[y\eta(q)^{-3}\Theta(y,q)]^2 - \\ - \frac{KJ}{2}[y\eta(q)^{-3}\Theta(y,q)]\mathrm{Ell}(\mathbb{P}^1, y, q) + \\ + \mathrm{Ell}(\mathbb{F}_k, y, q), \end{aligned} \tag{39}$$

where $J = pJ_1 + rJ_2$ and $K = -(k+2)J_1 - 2J_2$ is canonical bundle divisor.

Now we can see that an obvious generalization of (27) for the case of chiral de Rham complex twisted by line bundle $O(J)$ on two-dimensional complete smooth toric variety $\mathbb{P}_\Sigma$ is given by

$$\begin{aligned} \mathrm{Ell}_{J}(\mathbb{P}_\Sigma, y, q) = \frac{J^2}{2}[y\eta(q)^{-3}\Theta(y,q)]^2 - \\ - \frac{KJ}{2}[y\eta(q)^{-3}\Theta(y,q)]\mathrm{Ell}(\mathbb{P}^1, y, q) + \\ + \mathrm{Ell}(\mathbb{P}_\Sigma, y, q), \end{aligned} \tag{40}$$

where J is the divisor of a line bundle $O(J)$ and K is canonical bundle divisor of $\mathbb{P}_\Sigma$.

It is interesting to note here a similarity of the expression (40) to the expression for the Euler characteristic of a line bundle which is given by the Riemann–Roch formula for surfaces [12]. Moreover, in the limit $q \to 0$ and $y \to 1$ the formula (40) reproduce it exactly. So one can think of the expression (40) as a chiralization of or string generalization the Riemann–Roch formula for surfaces. In this more general situation we see again the infinite tower of open string contributions from J^2 $D0$-branes, one $D2$-brane wrapping the divisor J and one $D4$-brane wrapping $\mathbb{P}_\Sigma$. Thus we obtain the explaination of the chiral Riemann–Roch formula in terms of open string oscillator contributions of bound state of D-branes.

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