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Multi-matrix models and genus one amplitudes

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We study the genus one correlation numbers in the matrix models and the minimal Liouville gravity. We compute the torus partition function for the matrix models. We calculate the one- and two-point correlation numbers of the $(3,p)$ matrix models and $(3,p)$ Minimal Liouville gravity.

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The computations in the Liouville theory were a complicated problem since its introduction in 1981 [1, 2]. Recently some progress in this area was made in [3], where it was noticed that the problem of the computation of the correlation numbers could be simplify in the case of so-called Minimal Liouville gravity (MLG), which is a Liouville gravity with its matter sector being Minimal Model of conformal field theory (CFT) [4]. With use of higher equations of motion in the Liouville theory [5] three- and four-point correlation numbers for genus zero surfaces were determined by Belavin and Zamolodchikov in [3].

On the other hand another approach to fluctuating two-dimensional surfaces was proposed in 1990s [6–11]. It was based on the idea of approximation of a continuum two-dimensional surface with its discrete triangulation and the replacement of the functional integral over continuum surfaces with finite-dimensional ones. Since it was technically equivalent to integration over matrices this idea is commonly called the Matrix Model approach. It was believed to give the same answers on the same physical question as the Liouville theory does. But it was observed by Moore et al. in [12] that the connection between these theories is not such straightforward. It was pointed out that due to contact terms correlation functions of matrix models don't satisfy fusion rules of CFT. They also resolved this problem for one- and two-point correlation functions. Their work continued in [13, 14]. All the mentioned papers concerned amplitudes on the sphere. Extension of these results to the torus was done for $(2,p)$ models in [15]. Their results were confirmed by direct computations in the Liouville theory [16]. The aim of the present work is to extend these results to the case of the $(3,p)$ models: to find the genus one partition function in the $(3,p)$ matrix models and to calculate torus one- and two-point functions in $(3,p)$ MLG with the use of the exact form of the resonance relation, which was carried out in [14].

Let us make some remarks concerning the notation. The main result of our paper is made for the case of $(3,p)$ MLG, but some of the formulas presented for more general case of (q,p) MLG.

The matrix models approach appeared as a technical tool to calculate integrals over discrete 2-dimensional surfaces. As it was noticed in [17], the partition function of the matrix model corresponding to the (q,p) Minimal Gravity can be described as solution of higher Kortweg–de Vries (KdV) equation with special initial conditions called the Douglas string equation. We will use a reformulation of these equations in terms of the action principle introduced in [18]. Define

$$Q = d^q + \sum_{\alpha=1}^{q-1} u_\alpha(x) d^{q-\alpha-1}, \tag{1}$$

$$S[u_\alpha] = \mathrm{Res}\left(Q^{p/q+1} + \sum_{m=1}^{q-1}\sum_{n=1}^{p-1} \tau_{m,n} Q^{|pm-qn|/q}\right), \tag{2}$$

where residue gives the coefficient of d^{-1} taken with a minus sing and integrated over x. The constant factors $\tau_{m,n}$ are called times. The Douglas sting equation reads

$$\frac{\delta S[u_\alpha]}{\delta u_\alpha(x)} = 0, \tag{3}$$

where $\delta/\delta u_\alpha(x)$ is a variation with respect to $u_\alpha(x)$. These are differential equations for the functions $u_\alpha(x)$, from which they could be (in principle) determined. The partition function could be found by integrating the equation

$$\frac{\partial^2 Z}{\partial x^2} = u_1^*(x), \tag{4}$$

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where $u^*_\alpha(x)$ is a suitably chosen solution of (3). The correlation numbers in the matrix models are defined as

$$\langle O_{m_1n_1}O_{m_2n_2}\dots O_{m_Nn_N}\rangle = \\ = \frac{\partial^N Z}{\partial\tau_{m_1n_1}\dots\partial\tau_{m_Nn_N}}\Big|_{\tau_{1,2}=\tau_{2,1}=\dots=\tau_{q-1,p-1}=0}, \qquad (5)$$

in this limit all times except $\tau_{1,1}=\mu\neq 0$ must be set to zero.

The matrix models partition function in the double scaling limit can be expanded as [11, 19, 20]

$$Z[\varepsilon^{-2\delta_{m,n}/\gamma}\tau_{m,n}] = \sum_{h=0}^{\infty}\varepsilon^{2(h-1)}Z_h[\tau_{m,n}], \qquad (6)$$

where $\gamma = 1+p/q$, and gravitational dimensions equal

$$\delta_{m,n} = \frac{p+q-|pm-qn|}{2q}. \qquad (7)$$

Also, the factors $\varepsilon^{-2\delta_{m,n}/\gamma}$ in the left-hand side of (6) can be obtained by formal replacement $d/dx\to\varepsilon d/dx$. It could be proved with the use of the dimensional analysis.

Using these arguments one can find the genus one partition function as order ε^0 of Z which solves string equation with replacement $d/dx\to\varepsilon d/dx$. After this replacement ε must be seen as parameter in the equations (3), (4). All the function and variables must be also expanded in series in the parameter ε: $S[u_\alpha,\varepsilon] = \sum_{n=0}^{\infty}\varepsilon^n S^{(n)}[u_\alpha]$ and $u^*_\alpha(x,\varepsilon) = \sum_{n=0}^{\infty}\varepsilon^n u^{*\,(n)}_\alpha(x)$. The evaluation of the genus one partition function Z_1 could be found in the appendix. The result is

$$Z_1 = -\frac{1}{8}\log\det\frac{\delta^2 S^{(0)}[u_\alpha]}{\delta u_\alpha\delta u_\beta}\Big|_{u_\alpha=u^{*\,(0)}_\alpha}, \qquad (8)$$

where $S^{(0)}$ is the spherical sting action and $u^{*\,(0)}_\alpha$ is a solution for the spherical sting equations, α,β take values $1,2$, the determinant is computed with respect to α,β. This is one of the main results of our paper.

For the $(2,p)$ models the partition function was obtained in [15]

$$Z_1^{(2,p)} = -\frac{1}{12}\log\frac{\delta^2 S^{(0)}[u_1]}{(\delta u_1)^2}\Big|_{u_1=u^{*\,(0)}_1}. \qquad (9)$$

These two formulas give a natural conjecture for the general model torus partition function

$$Z_1^{(q,p)} = -\frac{q}{24}\log\det\frac{\delta^2 S^{(0)}}{\delta u_\alpha\delta u_\beta}\Big|_{u=u^{*\,(0)}_\alpha}, \qquad (10)$$

where α takes the values $1,...,q-1$. This conjecture differs by factor q from one made in [21].

Now, let's pass to the Liouville gravity. It's known, that any two-dimensional conformal field theory can be coupled to gravity. It was shown by Polyakov [1], that after choosing conformal gauge for the metric tensor $g_{ab}=e^{\varphi}\hat g_{ab}$, the full theory splits in two almost independent parts: initial CFT part, and Liouville theory of scalar field φ. For this reason, these theories are usually called Liouville gravity. If initial CFT is a minimal model of CFT [4] then one achieves model called Minimal Liouville gravity. We refers the reader for the additional information about the MLG to the other sources [22, 19].

The (q,p) minimal model of CFT [4] has central charge $c=1-(p-q)^2/6pq$. It consists of a finite number of primary fields $\Phi_{m,n}$ where $m=1,\dots,q-1$ and $n=1,\dots,p-1$. Because of reflection symmetry ($\Phi_{q-m,p-n}=\Phi_{m,n}$), only half of these fields are independent. In MLG these fields become "dressed"

$$O_{m,n} = \int\Phi_{m,n}e^{2b\delta_{m,n}\varphi(x)}\sqrt{\hat g}d^2x, \qquad (11)$$

where $b=\sqrt{q/p}$, integration is over 2-dimensional manifold and gravitational dimensions $\delta_{m,n}$ are the same as in (7).

In this article we will mostly be interested in deformed partition function

$$Z[\lambda] = \left\langle\exp\left(\sum_{(m,n)}\lambda_{m,n}O_{m,n}\right)\right\rangle. \qquad (12)$$

Correlation numbers of fields $O_{m,n}$ satisfy

$$\langle O_{m_1n_1}O_{m_2n_2}\dots O_{m_Nn_N}\rangle = \frac{\partial^N Z}{\partial\lambda_{m_1n_1}\dots\partial\lambda_{m_Nn_N}}\Big|_{\lambda_{m,n}=0}, \qquad (13)$$

in this limit all the couplings $\lambda_{m,n}$ should be set to zero.

Both the discrete and the continuous approaches are believed to provide the same answers to the similar questions, but as it was noticed in [12], there are some arbitrariness because of contact terms in the correlation functions. This obstacle can be resolve by the use of non- linear resonance transformation of the form

$$\tau_{m,n} = \lambda_{m,n} + \sum_{m_1,n_1}C_{m,n}^{(m_1,n_1)(m_2,n_2)}\lambda_{m_1,n_1}\lambda_{m_2,n_2}+\dots, \qquad (14)$$

where all the term satisfy resonance condition: sum of the gravitational dimension in every term must coincide.

In case $q=3$, the Kac table consist of two rows. Due to the reflection symmetry only half of these fields are independent. We will consider first row of the Kac table as independent variables. Also, we will label fields, times and coupling constants as $\Phi_k = \Phi_{1,k+1}$, $O_k = O_{1,k+1}$, $\tau_k = \tau_{1,k+1}$, and $\lambda_k = \lambda_{1,k+1}$ where $k = 0, \ldots, p-2$.

In the following, we are to calculate correlation numbers of $(3,p)$ models in both the matrix models and in the MLG. This is another result of our paper.

In the matrix models one should use the relation (5) for the computation of correlation number. The calculation is straightforward, so we will present only results. One must take the derivatives of (6) with respect to times $\tau_{m,n}$. There are two tricky points in it. The first point is the root of the string equations. As in [14] we will choose one special root. Namely, $u_2^{*(0)} = 0$. The second point is one should remember that root also depends on times $\tau_{m,n}$.

All the following formulas are given for $k_i < s$, where s is quotient of p divided by 3. We will use normalized $S^{(0)}[u_\alpha]$ and τ_k in the way that $\delta S^{(0)}/\delta u_1\Big|_{u_2=0}$ and $\delta S^{(0)}/\delta u_2\Big|_{u_2=0}$ will have coefficients 1 in all its terms.

For $k < s$ and p, k – even, we have

$$\langle O_k \rangle = \frac{1}{24}(p + 3k - 1)(-\mu)^{-k/2-1}. \tag{15}$$

For p – even and k – odd, one-point function is zero.

For two point correlation function we have

$$\langle O_{k_1} O_{k_2} \rangle = \frac{1}{48}\left[(5+p)(k_1+k_2) + 3(k_1^2 + k_2^2) + \right.$$
$$\left. + 3k_1k_2 + 2(p-1)\right](-\mu)^{(4+k_1+k_2)/2} \tag{16}$$

for p, k_i – even and $k_i < s$;

$$\langle O_{k_1} O_{k_2} \rangle = -\frac{1}{432} \times$$
$$\times (-1 + 3k_1 + p)(-1 + 3k_2 + p)(-\mu)^{-(k_1+k_2)/2-2} \tag{17}$$

for p – even, k_i – odd, and $k_i < s$;

$$\langle O_k \rangle = \frac{1}{24}(p + 3k - 1)(-\mu)^{-k/2-1} \tag{18}$$

for p – odd, k – even, and $k < s$. For two point correlation function we have

$$\langle O_{k_1} O_{k_2} \rangle = \frac{1}{48}\left[(5+p)(k_1+k_2) + 3(k_1^2 + k_2^2) + \right.$$
$$\left. + 3k_1k_2 + 2(p-1)\right)(-\mu)^{(4+k_1+k_2)/2} \tag{19}$$

for p – odd, k_i – even, and $k_i < s$. And

$$\langle O_{k_1} O_{k_2} \rangle = -\frac{1}{48} \times$$
$$\times (-2 + 3k_1 + p)(-2 + 3k_2 + p)(-\mu)^{-(k_1+k_2)/2-2} \tag{20}$$

for p – even, k_i – odd, and $k_i < s$.

Now, lets proceed to the MLG correlation numbers. The main issue in calculations of MLG correlators with use of matrix models results is to find all coefficients in relation (14) between $t_{m,n}$ and $\lambda_{m,n}$. The criteria for this is fusion rules in MLG which results in equation for coefficients in (14). Some first of them were determined in [14] and we will use their results in the following.

Computations in Liouville frame is quite the same to the matrix model ones. The main difference is that derivatives should be made with respect to $\lambda_{m,n}$. We will rescale $\lambda_{1,1} = \mu = 1$ for simplicity.

The results for the zero and one-point correlation numbers are

$$\langle 1 \rangle = -\frac{1}{4}\log p, \tag{21}$$

$$\langle O_k \rangle = -\frac{2 + 6k + 3k^2 - 2(k+1)p}{16p}, \tag{22}$$

where k, p – even numbers. One-point function for k – odd is zero. The two point correlation numbers are

$$\langle O_{k_1} O_{k_2} \rangle = -\frac{1}{32p^2}\{9k_1^3(1+k_2) +$$
$$+ 9k_1^2(1+k_2)(4+k_2) + 3(2+k_2)[2 + 3k_2(2+k_2)] +$$
$$+ 3k_1(1+k_2)[14 + 3k_2(4+k_2)] -$$
$$- 6(1+k_1)(1+k_2)(2+k_1+k_2)p\}, \tag{23}$$

where k_1, k_2, p – even numbers.

All these results depend on the normalisation of fields and partition function. Since relative normalisation of the fields in our paper is the same with ones in [14], we can introduce the normalisation independent quantities

$$\sqrt{\frac{Z_0^{\text{Sphere}}}{Z_{kk}^{\text{Sphere}}}}\frac{Z_k^{\text{Torus}}}{Z_0^{\text{Torus}}} = \sqrt{\frac{p[p-3(k+1)]}{(p+3)(p-3)}} \times$$
$$\times \frac{2 + 6k + 3k^2 - 2(k+1)p}{4p\log p}, \tag{24}$$

$$\frac{Z_0^{\text{Sphere}}}{\sqrt{Z_{k_1k_1}^{\text{Sphere}} Z_{k_2k_2}^{\text{Sphere}}}}\frac{Z_{k_1k_2}^{\text{Torus}}}{Z_0^{\text{Torus}}} =$$
$$= \sqrt{\frac{p^2[p-3(k_1+1)][p-3(k_2+1)]}{(p+3)^2(p-3)^2}}, \tag{25}$$

$$\frac{1}{8p^2\log p}\{9k_1^3(1+k_2) + 9k_1^2(1+k_2)(4+k_2) +$$
$$+ 3(2+k_2)[2 + 3k_2(2+k_2)] +$$
$$+ 3k_1(1+k_2)[14 + 3k_2(4+k_2)] -$$
$$- 6(1+k_1)(1+k_2)(2+k_1+k_2)p\}.$$

To summarize, we calculated the torus partition function (6) and the one- and two-point functions in the MLG and matrix model frames for the case of $(3,p)$ models. The result obtained for the MLG is indirect, but the same method for the $(2,p)$ models was carried in [15] and then was proved by direct computation of [16].

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Appendix. Approximate solution of string equation. Because the full solution is quite lengthy we will only point out the main points of the calculation.

The first problem is to find the string action up to ε^2 order. One can do it using the following technique [19, 20]. The idea is to find the recurrence relation for residues of the form $\operatorname{Res} Q^{l+\alpha/3}$. Parametrising the negative power part as $Q_-^{l+\alpha/3} = \{R_l, d^{-1}\} + \{K_l, d^{-2}\} + \{M_l, d^{-3}\} + \ldots$, using the fact that any operator commutes with any its power and also using relation $Q^{l+\alpha/3} = Q_-^{l+\alpha/3} + Q_+^{l+\alpha/3}$, we have

$$[Q_+^{l+\alpha/3}, Q] = [Q, Q_-^{l+\alpha/3}]. \tag{26}$$

Notice that the left-hand side contains only non-negative powers of d, and right-hand side contains only finite number of non-negative powers of d. They are

$$[Q_+^{l+\alpha/3}, Q] = [Q, Q_-^{l+\alpha/3}] = 6R_l'd + 6K_l' + 3R_l''. \tag{27}$$

By simple computation one can find

$$Q_+^{l+1+\alpha/3} = Q^{l-2/3}Q + 2R_l d^2 + (2K_l - R_l')d + + 2M_l + R_l'' - 2K_l' + 2R_l u. \tag{28}$$

Commutating both side with Q, we obtain three recurrence equations on three unknown functions K_l, M_l, and R_l. We solved this equation up to the order ε^2. Using this solution and definition (2) we found string action $S[u_\alpha, \varepsilon] = S^{(0)}[u_\alpha] + \varepsilon S^{(1)}[u_\alpha] + \varepsilon^2 S^{(2)}[u_\alpha] + O(\varepsilon^2)$ up to the second order in epsilon. Our next step was to solve Douglas Eq. (3) which could be represented as $u_1^*(x,\varepsilon) = u_1^{*(0)}(x) + \varepsilon^2 u_1^{*(2)}(x) + O(\varepsilon^2)$ and important point here is that $u_1^{*(2)}$ can be represented using only $S^{(0)}[u_\alpha]$ and its partial derivatives with respect to u_1, u_2. The last step is to integrate (4) with u_* replaced with $u_1^{*(2)}$. The result is (6).

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