

Generalized Jack polynomials and the AGT relations for the $SU(3)$ group

S. Mironov^{+*1)}, A. Morozov^{+×1)}, Y. Zenkevich^{+*1)}

Institute for Theoretical and Experimental Physics, 117218 Moscow, Russia

**Institute for Nuclear Research of the RAS, 142190 Troitsk, Russia*

×Department of Physics, Lomonosov MSU, 119991 Moscow, Russia

Submitted 18 December 2013

We find generalized Jack polynomials for the $SU(3)$ group and verify that their Selberg averages for several first levels are given by Nekrasov functions. To compute the averages we derive recurrence relations for the \mathfrak{sl}_3 Selberg integrals.

DOI: 10.7868/S0370274X14020106

1. Introduction. The AGT relations [1] provide an extremely interesting link between the four dimensional $\mathcal{N} = 2$ gauge theories of class S [2] and two dimensional conformal field theories. Moreover, these relations offer a new view on a variety of related fields, such as integrable systems [3], matrix models [4, 5], etc.

The most surprising property of the AGT relations is that they give an additional and unexpected structure on the conformal field theory Hilbert space. One usually writes the states in this space as descendants of some primary field: $L_{-Y}|\alpha\rangle$, all the correlators therefore become sums over Young diagrams Y . However, the Nekrasov function is a sum over *pairs* of Young diagrams \mathbf{Y} . Finding the corresponding basis $|\mathbf{Y}\rangle$ in CFT is an interesting problem. This basis was found explicitly in the case of $c = 1$ [6] and it was argued to exist in the general case [7]. Concretely, if one performs the bosonization of the Virasoro algebra, the basis vectors are expressed through the generalized Jack polynomials which are defined as the polynomial eigenfunctions of a certain differential operator [8]. (It is suspected that generalized Jack polynomials are actually common eigenfunctions of an infinite family of differential operators.)

One can also compute the correlators in CFT using the Dotsenko–Fateev approach, in which they are given by certain multiple integrals related to the Selberg integrals [9]. In this setting the generalized Jack polynomials also play a distinguished role: their Selberg averages are factorised into a product of linear functions of momenta. More precisely, the averages are given by the Nekrasov functions.

More generally, the AGT relations map the gauge theory with the $SU(N)$ group to the four point conformal block in the Toda field theory with two general and two degenerate fields [10]. The same factorisation of the \mathfrak{sl}_N Selberg averages should happen in this case as well.

In this Letter we explicitly find generalized Jack polynomials for the group $SU(3)$ and check that their Selberg averages indeed reproduce the Nekrasov functions on the first levels. To compute the averages we derive the W -constraints for the β -deformed A_2 quiver matrix model. In Section 2 we introduce the differential operator whose polynomial eigenfunctions are given by the generalized Jack polynomials, compute them explicitly and check their elementary properties. In section 3 we derive the Dotsenko–Fateev representation of the conformal block in Toda field theory and show that the AGT relations hold if certain Selberg averages of generalised Jack polynomials are given by the Nekrasov functions. Using the W constraints we compute the averages and check the relevant formulas for the first levels. The Nekrasov functions and AGT relations are provided in Appendix A. The W constraints are presented in Appendix B.

2. Differential operator. Generalized Jack polynomials $J_\lambda(\{p_k^{(a)}\}|\beta, \{a_a\})$ are labelled by an N -tuple of Young diagrams $\lambda = \{\lambda_{(1)}, \lambda_{(2)}, \dots\}$. They are eigenfunctions of the differential operator $\hat{D} = \sum_{a=1}^N \hat{H}_a + \sum_{a<b} \hat{H}_{ab}$, where

$$2\hat{H}_a = \sum_{n,m \geq 0} \left[\beta(n+m) p_n^{(a)} p_m^{(a)} \frac{\partial}{\partial p_{n+m}^{(a)}} + nm p_{n+m}^{(a)} \frac{\partial^2}{\partial p_n^{(a)} \partial p_m^{(a)}} \right] +$$

¹⁾e-mail: sa.mironov_1@physics.msu.ru; andrey.morozov@itep.ru; yegor.zenkevich@gmail.com

$$+ \sum_{n \geq 1} [2a_a + (1 - \beta)(n - 1)] n p_n^{(a)} \frac{\partial}{\partial p_n^{(a)}},$$

$$\hat{H}_{ab} = (1 - \beta) \sum_{n \geq 1} n^2 p_n^{(b)} \frac{\partial}{\partial p_n^{(a)}}$$

with eigenvalues $\varkappa_{\lambda} = \sum_{a=1}^N \sum_{(i,j) \in \lambda_a} [a_a - \beta(i - 1) + (j - 1)]$.

This definition allows one to find first few polynomials as linear combinations of $p_k^{(a)}$. Since the eigenvalues \varkappa_{λ} are known, the problem of finding the eigenfunctions reduces to a system of linear algebraic equations. The elementary properties of the generalized Jack polynomials are:

1) *Orthogonality and normalization.* If one sets $\tilde{J}_{\mathbf{A}}(p) = \prod_{a=1}^N m_{A_a}(p) + \dots$ with m_A being the monomial symmetric function then the family of generalized Jack polynomials is orthonormal, $\langle \tilde{J}_{\mathbf{A}}^*, \tilde{J}_{\mathbf{B}} \rangle = \delta_{\mathbf{A}\mathbf{B}}$, with respect to the scalar product

$$\langle f(p_k), g(p_k) \rangle = f \left(\frac{n}{\beta} \frac{\partial}{\partial p_k} \right) g(p_k) \Big|_{p_k=0}.$$

Here the conjugate polynomials are defined as follows:

$$\begin{aligned} \tilde{J}_{A_1, A_2, \dots, A_N}^*(p^{(1)}, \dots, p^{(N)} | \beta, a_1, \dots, a_N) = \\ = \tilde{J}_{A_N, A_{N-1}, \dots, A_1}^*(p^{(N)}, \dots, p^{(1)} | \beta, a_N, \dots, a_1). \end{aligned} \quad (1)$$

It will be more convenient for us to use a different normalization

$$J_{\mathbf{A}}(p) = (-1)^{|\mathbf{A}|} \beta^{-2|\mathbf{A}|} \prod_{a < b} g_{A_b, A_a}(a_b - a_a) \tilde{J}_{\mathbf{A}}(p), \quad (2)$$

where g_{AB} are given in Appendix A. In this case the norms of the Jack polynomials are given by the vector parts of the Nekrasov functions:

$$\langle J_{\mathbf{A}}^*, J_{\mathbf{B}} \rangle = \beta^{-4|\mathbf{A}|} z_{\text{vect}}(\mathbf{A}, \mathbf{a}) \delta_{\mathbf{A}, \mathbf{B}}; \quad (3)$$

2) *The “inversion” relation.* This relation can be derived by taking the adjoint of the differential operator \hat{D} :

$$J_{\mathbf{A}} \left(-\frac{p_k^{(a)}}{\beta} \Big| \beta, \mathbf{a} \right) = (-\beta)^{(N-4)|\mathbf{A}|} J_{\mathbf{A}^T} \left(p_k^{(a)} \Big| \frac{1}{\beta}, -\frac{\mathbf{a}}{\beta} \right); \quad (4)$$

3) *Cauchy completeness identity.*

$$\sum_{\mathbf{A}} \beta^{4|\mathbf{A}|} \frac{J_{\mathbf{A}}^*(p_k^{(a)}) J_{\mathbf{A}}(q_k^{(a)})}{z_{\text{vect}}(\mathbf{A}, \mathbf{a})} = \exp \left(\beta \sum_{k \geq 1} \sum_{a=1}^N \frac{p_k^{(a)} q_k^{(a)}}{k} \right). \quad (5)$$

For $\beta \rightarrow 1$ the operator \hat{D} becomes the cut-and-join operator [10] and the generalized Jack polynomials factorize into products of the Schur polynomials

$J_{\lambda}(\{p_k^{(a)}\} | \beta \rightarrow 1, \{a_a\}) = \prod_{a=1}^N s_{\lambda^{(a)}}(\{p_k^{(a)}\})$ (Note also that for $\beta \neq 1$ the generalized Jack polynomials do not factorize into the Jack polynomials (as suggested in [12]), but are instead linear combinations thereof).

We were able to find and check the properties of the generalized Jack polynomials for $N = 3$ up to level 3. We list here the polynomials at the level 1:

$$\begin{aligned} J_{\{\{1\}, \{1\}, \{1\}\}} &= \frac{(a_1 - a_2)(a_3 - a_1)p_1^{(1)}}{\beta} + \\ &+ \frac{(\beta - 1)(a_1 - a_3)p_1^{(2)}}{\beta} + \frac{(1 - \beta)(\beta - 1 - a_1 + a_2)p_1^{(3)}}{\beta}, \end{aligned}$$

$$\begin{aligned} J_{\{\{1\}, \{1\}, \{1\}\}} &= \frac{(a_3 - a_2)(\beta + a_1 - a_2 - 1)p_1^{(2)}}{\beta} + \\ &+ \frac{(\beta - 1)(\beta + a_1 - a_2 - 1)p_1^{(3)}}{\beta}, \end{aligned}$$

$$J_{\{\{1\}, \{1\}, \{1\}\}} = -\frac{(\beta + a_1 - a_3 - 1)(\beta + a_2 - a_3 - 1)p_1^{(3)}}{\beta},$$

$$J_{\{\{1\}, \{1\}, \{1\}\}}^* = \frac{(\beta - a_1 + a_2 - 1)(\beta - a_1 + a_3 - 1)p_1^{(1)}}{\beta},$$

$$\begin{aligned} J_{\{\{1\}, \{1\}, \{1\}\}}^* &= -\frac{(\beta - 1)(\beta - a_2 + a_3 - 1)p_1^{(1)}}{\beta} - \\ &- \frac{(a_1 - a_2)(\beta - a_2 + a_3 - 1)p_1^{(2)}}{\beta}, \end{aligned}$$

$$\begin{aligned} J_{\{\{1\}, \{1\}, \{1\}\}}^* &= \frac{(\beta - 1)(\beta + a_2 - a_3 - 1)p_1^{(1)}}{\beta} + \\ &+ \frac{(\beta - 1)(a_1 - a_3)p_1^{(2)}}{\beta} + \frac{(a_1 - a_3)(a_2 - a_3)p_1^{(3)}}{\beta}. \end{aligned}$$

3. Factorisation of DF integrals. The free field representation of the four point conformal block in Toda field theory is given by the Dotsenko–Fateev integrals:

$$\begin{aligned} \mathcal{B} &= (1 - q)^{(\alpha_2 \alpha_3)} \left\langle \left\langle \prod_{a=1}^{N-1} \left[\prod_{i=1}^{n_+^a} (1 - qx_i^{(a)})^{v_+^a} \times \right. \right. \right. \\ &\times \left. \prod_{j=1}^{n_-^a} (1 - qy_j^{(a)})^{v_-^a} \prod_{i=1}^{n_+^a} \prod_{j=1}^{n_-^a} (1 - qx_i^{(a)} y_j^{(a)})^{2\beta} \right] \times \\ &\times \prod_{a=1}^{N-2} \prod_{i=1}^{n_+^{a+1}} \prod_{j=1}^{n_-^{a+1}} (1 - qx_i^{(a)} y_j^{(a+1)})^{-\beta} \times \\ &\times \left. \left. \left. \prod_{a=1}^{N-2} \prod_{i=1}^{n_+^{a+1}} \prod_{j=1}^{n_-^{a+1}} (1 - qx_i^{(a+1)} y_j^{(a)})^{-\beta} \right] \right\rangle \right\rangle, \quad (6) \end{aligned}$$

where the intermediate momentum is determined by the screening charges: $\alpha = \alpha_1 + \alpha_2 + 2b \sum_{a=1}^N n_+^a e_a$. The central charge is $c = (N-1)[1 - N(N+1)Q^2]$ and u_{\pm}^a , v_{\pm}^a are given in Appendix A. The \mathfrak{sl}_N Selberg averages are defined as follows:

$$\begin{aligned} \langle f(x) \rangle_{\pm} = & \frac{1}{S} \int_{\mathcal{C}} \prod_{a=1}^{N-1} \left[d^{m_{\pm}^a} x^{(a)} \prod_{i=1}^{n_{\pm}^a} (x_i^{(a)})^{u_{\pm}^a} (x_i^{(a)} - 1)^{v_{\pm}^a} \times \right. \\ & \times \left. \prod_{1 \leq i < j \leq n_{\pm}^a} (x_i^{(a)} - x_j^{(a)})^{2\beta} \right] \times \\ & \times \prod_{a=1}^{N-2} \prod_{i=1}^{n_{\pm}^a} \prod_{j=1}^{n_{\pm}^{a+1}} (x_i^{(a)} - x_j^{(a+1)})^{-\beta} f(x), \end{aligned} \quad (7)$$

and S is the integral without insertions. The AGT relation requires the dimensions of the primary fields at $z = 1$ and q to be maximally degenerate [10]. This implies that $v_+^a = v_+ \delta_1^a$, $v_-^a = v_- \delta_{N-1}^a$ in the Selberg average. For these parameters the expression under the correlator can be nicely written as an exponential:

$$\begin{aligned} \mathcal{B} = & (1-q)^{(\alpha_2 \alpha_3)} \left\langle \left\langle \exp \left\{ \beta \sum_{k>0} \frac{q^k}{k} \times \right. \right. \right. \\ & \times \left[\sum_{a=1}^{N-2} (p_k^{(a+1)} - p_k^{(a)}) (q_k^{(a)} - q_k^{(a+1)}) + \right. \\ & + p_k^{(N-1)} \left(-q_k^{(N-1)} - \frac{v_-}{\beta} \right) + \\ & \left. \left. \left. + \left(-p_k^{(1)} - \frac{v_+}{\beta} \right) q_k^{(1)} \right] \right\} \right\rangle_+ \right\rangle_-, \end{aligned} \quad (8)$$

where $p_k^{(a)} = \sum_{i=1}^{n_+^a} (x_i^{(a)})^k$, $q_k^{(a)} = \sum_{j=1}^{n_-^a} (y_j^{(a)})^k$.

Using the Cauchy completeness identity (5) one obtains the sum of factorised ‘‘plus’’ and ‘‘minus’’ correlators:

$$\begin{aligned} Z = & \sum_{\mathbf{A}} \frac{q^{|\mathbf{A}|} \beta^{4|\mathbf{A}|}}{z_{\text{vect}}(\mathbf{A}, a)} \times \\ & \times \left\langle J_{\mathbf{A}} \left(-p_k^{(1)} - \frac{v_+}{\beta}, p_k^{(2)} - p_k^{(1)}, \dots, p_k^{(N-1)} \right) \right\rangle_+ \times \\ & \times \left\langle J_{\mathbf{A}}^* \left(q_k^{(1)}, q_k^{(1)} - q_k^{(2)}, \dots, -q_k^{(N-1)} - \frac{v_-}{\beta} \right) \right\rangle_-. \end{aligned}$$

To check the AGT conjecture one should ensure that the Selberg averages of the generalised Jack polynomials reproduce the individual terms in the Nekrasov function (11):

$$\begin{aligned} \left\langle J_{\mathbf{A}} \left(-p_k^{(1)} - \frac{v_+}{\beta}, p_k^{(2)} - p_k^{(1)}, \dots, p_k^{(N-1)} \right) \right\rangle_+ & \stackrel{?}{=} \\ & \stackrel{?}{=} \prod_{i=1}^N \prod_{f=1}^N f_{A_i}(m_f + a_i), \end{aligned} \quad (9)$$

$$\begin{aligned} \left\langle J_{\mathbf{A}}^* \left(q_k^{(1)}, q_k^{(1)} - q_k^{(2)}, \dots, -q_k^{(N-1)} - \frac{v_-}{\beta} \right) \right\rangle_- & \stackrel{?}{=} \\ & \stackrel{?}{=} \prod_{i=1}^N \prod_{f=N}^{2N} f_{A_i}(m_f + a_i). \end{aligned} \quad (10)$$

To compute the Selberg averages for several lowest diagrams one should employ the Virasoro and W constraints [13]. Using this method we have checked the relations above for $N = 3$ and diagrams up to order two. We derive the necessary constraints in Appendix B.

4. Conclusions and outlook. In this Letter we have explicitly found the generalised Jack polynomials for the group $SU(3)$ and checked the AGT relations for the \mathfrak{sl}_3 Selberg averages on several first levels. We have also derived the W constraints for the β -deformed quiver matrix model, which provide recurrence relations for the correlators and in principle allow the computation at the arbitrary level.

It would be interesting to investigate the connection between W constraints for the β -ensembles and the family of the Jack commuting differential operators (of which \hat{D} is only one example). It also seems plausible that the general form of these operators can be found along the lines of [14].

We would like to thank Al. Morozov and A. Mironov for helpful discussions. A.M. and Y.Z. acknowledge the hospitality of the International Institute of Physics and DFTE-UFRN in Natal, Brazil where part of this work was done. Our work is partly supported by Ministry of Education and Science of the Russian Federation under contract 8410 (S. M.), by RFBR grants 13-02-00478, 12-01-33071 mol_a_ved (Y. Z.), 13-02-91371-ST, 12-02-92108-Yaf_a, 11-01-00962 (A. M.), 12-01-00525 (S. M.) and the Dynasty Foundation (S. M. and A. M.).

Appendix A. Nekrasov functions and AGT relations. The Nekrasov partition function for the $SU(N)$ theory with $N_f = 2N$ fundamental hypermultiplets is given by

$$Z_{\text{Nek}} = \sum_{\mathbf{A}} q^{|\mathbf{A}|} \frac{\prod_{i=1}^N \prod_{f=1}^{2N} f_{A_i}(m_f + a_i)}{z_{\text{vect}}(\mathbf{A}, \mathbf{a})}, \quad (11)$$

where $f_A(x) = \prod_{(i,j) \in A} [x + \epsilon_1(i-1) + \epsilon_2(j-1)]$, $z_{\text{vect}}(\mathbf{A}, \mathbf{a}) = \prod_{i,j=1}^N g_{A_i A_j}(a_i - a_j)$, and

$$g_{AB}(x) = \prod_{s \in A} [x + \epsilon_1 \text{Arm}_A(s) - \epsilon_2 \text{Leg}_B(s) + \epsilon_1] \times \\ \times [x + \epsilon_1 \text{Arm}_A(s) - \epsilon_2 \text{Leg}_B(s) - \epsilon_2].$$

The AGT relations for $N = 3$ are:

$$b(\alpha_1 \cdot e_a) = u_+^a = m_a - m_{a+1} - 1 + \beta, \\ b(\alpha_3 \cdot e_a) = v_-^a = -\delta_a^{N-1} \sum_{b=1}^N \tilde{m}_b, \\ b(\alpha_4 \cdot e_a) = u_-^a = \tilde{m}_a - \tilde{m}_{a+1} - 1 + \beta, \\ b(\alpha_2 \cdot e_a) = v_+^a = -\delta_a^1 \sum_{b=1}^N m_b, \\ \beta n_+^1 = a_2 + a_3 + m_2 + m_3, \\ \beta n_+^2 = a_3 + m_3,$$

where $\tilde{m}_a = m_{N+a}$ and $a = 1, 2$. Masses m_a , the vev a_i and $\epsilon_{1,2}$ have all the dimension of mass. In this paper we set the overall mass scale so that $\epsilon_1 = -b^2$, $\epsilon_2 = 1$. In the calculations involving the Jack polynomials we also use the parameter $\beta = b^2$.

Appendix B. W_3 and Virasoro constraints for $s\mathfrak{L}_3$ Selberg averages. The Virasoro constraints for the Selberg integral are written as follows

$$\left\langle \left[\hat{A}(z) - \hat{B}(z) - \beta \rho_1(z) \rho_2(z) \right] \Phi(w, \bar{w}) \right\rangle = 0, \quad (12)$$

$$\text{where } \hat{A}(z) \Phi(w, \bar{w}) = \left\{ (\beta - 1) \partial_z \rho_1(z) + \beta \rho_1^2(z) + \right. \\ \left. + \frac{u_1 \rho_1(z)}{z} + \frac{v_1 \rho_1(z)}{z-1} - \frac{u_1 \rho_1(0)}{z} - \frac{v_1 \rho_1(1)}{z-1} + \right. \\ \left. + \sum_{l=1}^m \frac{1}{\rho_1(w_l)} \partial_{w_l} \left[\frac{\rho_1(z) - \rho_1(w_l)}{z - w_l} \right] \right\} \Phi(w, \bar{w}), \\ \hat{B}(z) \Phi(w, \bar{w}) = \\ = \left\{ (1 - \beta) \partial_z \rho_2(z) - \beta \rho_2^2(z) - \frac{u_2 \rho_2(z)}{z} + \frac{u_2 \rho_2(0)}{z} - \right. \\ \left. - \sum_{l=1}^{\bar{m}} \frac{1}{\rho_2(\bar{w}_l)} \partial_{\bar{w}_l} \left(\frac{\rho_2(z) - \rho_2(\bar{w}_l)}{z - \bar{w}_l} \right) \right\} \Phi(w, \bar{w}),$$

and $\rho_a(z) = \sum_{i=1}^{N_a} \frac{1}{z - x_i^{(a)}}$, $\Phi(w, \bar{w}) = \prod_{k=1}^m \rho_1(w_k) \times \prod_{\bar{k}=1}^{\bar{m}} \rho_2(\bar{w}_{\bar{k}})$. The W constraints are given by

$$\left\langle \left\{ \beta \rho_1(z) \rho_2(z) [\rho_1(z) - \rho_2(z)] + (\beta - 1) \rho_2(z) \partial_z \rho_1(z) + \right. \right. \\ \left. \left. + \frac{\beta - 1}{\beta} \partial_z \hat{B}(z) + \left[\frac{u_1 + u_2}{\beta z} + \frac{v_1}{\beta(z-1)} \right] \hat{B}(z) + \right. \right. \\ \left. \left. + \left(\frac{u_1}{z} + \frac{v_1}{z-1} \right) \rho_1(z) \rho_2(z) + \frac{1}{\beta} \sum_{l=1}^m \frac{1}{\rho_1(w_l)} \partial_{w_l} \times \right. \right. \\ \left. \left. \times \left[\frac{\hat{B}(z) + \beta \rho_1(z) \rho_2(z) - \hat{B}(w_l) - \beta \rho_1(w_l) \rho_2(w_l)}{z - w_l} \right] + \right. \right. \\ \left. \left. + \frac{1}{\beta} \sum_{\bar{l}=1}^{\bar{m}} \frac{1}{\rho_2(\bar{w}_{\bar{l}})} \partial_{\bar{w}_{\bar{l}}} \left[\frac{\hat{B}(z) - \hat{B}(\bar{w}_{\bar{l}})}{z - \bar{w}_{\bar{l}}} \right] - \frac{u_1}{\beta z} \hat{A}(0) - \right. \right. \\ \left. \left. - \frac{u_2}{\beta z} \hat{B}(0) - \frac{v_1}{\beta(z-1)} \hat{A}(1) \right\} \Phi(w, \bar{w}) \right\rangle = 0. \quad (13)$$

If one considers the expansion of the Virasoro and W constraints z^{-1} , w^{-1} , \bar{w}^{-1} one obtains the recurrence relations for the correlators of $p_k^{(a)}$. The coefficients in front of z^{-1} , z^{-2} , z^{-3} fix the correlators of $\rho_{1,2}(0)$, $\rho_1(1)$, $\hat{A}(0)$, $\hat{B}(0)$, and $\hat{A}(1)$.

1. L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Lett. Math. Phys.* **91**, 167 (2010); arXiv:0906.3219 [hep-th].
2. D. Gaiotto, *JHEP* **1208**, 034 (2012) arXiv:0904.2715 [hep-th].
3. A. Gorsky, I. Krichever, A. Marshakov, A. Mironov, and A. Morozov, *Phys. Lett. B* **355**, 466 (1995); R. Donagi and E. Witten, *Nucl. Phys. B* **460**, 299 (1996); arXiv:hep-th/9510101; N. A. Nekrasov and S. L. Shatashvili, arXiv:0908.4052 [hep-th]; A. Mironov and A. Morozov, *JHEP* **1004**, 040 (2010); arXiv:0910.5670 [hep-th]; A. Mironov and A. Morozov, *J. Phys. A* **43**, 195401 (2010); [arXiv:0911.2396 [hep-th]; A. Mironov, A. Morozov, Y. Zenkevich, and A. Zotov, *JETP Lett.* **97**, 45 (2013); arXiv:1204.0913 [hep-th]; A. Mironov, A. Morozov, B. Runov, Y. Zenkevich, and A. Zotov, *Lett. Math. Phys.* **103**(3), 299 (2013); arXiv:1206.6349 [hep-th]; A. Mironov, A. Morozov, B. Runov, Y. Zenkevich, and A. Zotov, *JHEP* **1312**, 034 (2013); [arXiv:1307.1502].
4. R. Dijkgraaf and C. Vafa, arXiv:0909.2453 [hep-th]; H. Itoyama, K. Maruyoshi and T. Oota, *Prog. Theor. Phys.* **123**, 957 (2010); arXiv:0911.4244 [hep-th]; T. Eguchi and K. Maruyoshi, *JHEP*, 07:081, 2010; T. Eguchi and K. Maruyoshi, (YITP-09-94), 2010; R. Schiappa and N. Wyllard, *J. Math. Phys.* **51**, 082304 (2010); arXiv:0911.5337 [hep-th]; A. Mironov, A. Morozov, and Sh. Shakirov, *JHEP*, 02:030, 2010; A. Mironov, A. Morozov, and Sh. Shakirov, *Int. J. Mod. Phys. A* **25** 3173 (2010); A. Mironov, A. Morozov, and And. Morozov, *Nucl. Phys. B* **843**, 534 (2011).
5. A. Mironov, A. Morozov and S. Shakirov, *JHEP* **1102**, 067 (2011); arXiv:1012.3137 [hep-th]; A. Mironov,

- A. Morozov, S. Shakirov and A. Smirnov, Nucl. Phys. B **855**, 128 (2012); arXiv:1105.0948.
6. A. Belavin and V. Belavin, Nucl. Phys. B **850**, 199 (2011); arXiv:1102.0343 [hep-th].
7. V.A. Alba, V.A. Fateev, A.V. Litvinov, and G.M. Tarnopolskiy, Lett. Math. Phys. **98**, 33 (2011); arXiv:1012.1312 [hep-th]; V.A. Fateev and A.V. Litvinov, JHEP **1201**, 051 (2012); arXiv:1109.4042 [hep-th].
8. A. Morozov and A. Smirnov, arXiv:1307.2576 [hep-th].
9. K.W.J. Kadell, Adv. Math. **130**, 33 (1997); K.W.J. Kadell, Compositio Math. **87**, 5 (1993); J.-E. Bourguine, JHEP, (08):046, 2012.
10. N. Wyllard, JHEP **0911**, 002 (2009); arXiv:0907.2189 [hep-th]; A. Mironov and A. Morozov, Nucl. Phys. B **825**, 1 (2010); arXiv:0908.2569 [hep-th].
11. A. Mironov, A. Morozov, and S. Natanzon, Theor. Math. Phys. **166**, 1 (2011); arXiv:0904.4227 [hep-th].
12. H. Zhang and Y. Matsuo, JHEP **1112**, 106 (2011); arXiv:1110.5255 [hep-th].
13. H. Itoyama and T. Oota, Nucl. Phys. B **852**, 336 (2011); arXiv:1106.1539 [hep-th]; H. Itoyama, T. Oota, R. Yoshioka, arXiv:1308.2068.
14. M. Nazarov and E. Sklyanin, SIGMA **9**, 078 (2013); arXiv:1309.6464 [nlin.SI]