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Localization of edge electrons in a 2D topological insulator strip

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The edge states in an ideal topological insulator are topologically protected. The same-spin edge states propagate in opposite directions on different sides of the strip and do not mix by tunneling. The interaction with impurities results in interedge same-spin backscattering. At low temperatures, the localization occurs and the conductance of a long wire vanishes. We study this localization in a numerical model of localized scatterers. The localization length is found to be exponentially long with respect to the strip width. The intraedge inelastic forward scattering destroys the coherence. These processes caused by phonons were explored. The transition temperature between kinetic and localization behaviors has been found.

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Introduction. Topological insulator (TI) is the novel object of solid state physics. It attracted a significant interest and gave birth to a wide publication activity (see, e.g., reviews [1, 2] and references therein; more recent references see in [3]). Topologically insulated states appear near the edges of a TI macroscopic sample. As a result of topological invariance, the edge states are insulated from the backscattering by spin conservation. Such macroscopic system has a quantized conductance. The situation does not change in an ideal (clean) strip. In a narrow strip overlapping of the edge states mixes the states with the same momentum and spin on different strip sides with opposite directions of motion. Edge state electrons with a fixed energy and momentum move in a fixed direction preferably along one edge of the sample. At the same time, the presence of disorder leads to backscattering and establishment of a kinetic conductivity [3] at a finite temperature. However, the low temperature situation has not been studied yet. It is apparent that the presence of conservative disorder will lead to the phase-coherent accumulation of backscattering events and, as a result, to the localization of the edge states.

The present paper is a continuation of our previous paper [3] where the backscattering on impurities, phonons and edge imperfectness was considered within framework of the kinetic equation. This approach is valid at a finite temperature when the phonon decoherence destroys the localization of the electron states. *Vice versa*, at zero temperature, the localization occurs, the studying of it is the purpose of the present paper. Besides, we shall consider dephasing due to the forward scattering by phonons, which determines the threshold between the kinetic and localization regimes. We shall find the localization length of the narrow strip of 2D TI with impurities and the temperature of transition between the kinetic and localization regimes.

Localization at low temperature. Here we shall study the edge electron states in a narrow strip of the 2D topological insulator HgTe with impurities at low temperatures. In presence of phase coherence the quantum approach based on the localization theory [4] is needed.

We deal with the 2D HgTe strip in the (x, y)-plane $-\mathcal{L}/2 < x < \mathcal{L}/2$, $-L/2 < y < L/2$, $\mathcal{L} \gg L$ with zero conditions for the wave functions at $y = \pm L$ and cyclic conditions at $x = \pm\mathcal{L}$. Our study is based on the effective 2×2 edge-states Hamiltonian

$$\hat{H} = -iv\hat{\sigma}_z\partial_x + \hat{U}(x). \qquad (1)$$

Here $\hat{U}(x) = \sum_n \hat{u}_n(x - x_n)$ is a 2×2 matrix of the total potential, $\hat{u}_n(x - x_n)$ are potentials of individual impurities situated in points (x_n, y_y). The matrix elements of $\hat{U}(x)$ and $\hat{u}_n(x - x_n)$ are given by a projection of the potential $U(x, y) = \sum_n u(x - x_n, y - y_n)$ onto the transversal wave functions of edge states $\psi_\sigma(y)$ (see Eqs. (2), (3) from [3]).

In the absence of the potential, the solutions of the stationary Schrödinger equation $(\hat{H} - E)\xi = 0$ are $\xi_{k,\sigma} = e^{ikx}(1 + \sigma, 1 - \sigma)/2$ with energy $E_{k,\sigma} = \sigma vk$.

The matrix of potential $\hat{U}(x)$ mixes the states of different edges. With account for the diagonal elements of the potential $U_{\sigma,\sigma}(x)$ only, the stationary solutions with the same $E_{k,\sigma}$ convert to

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$$\xi_{k,\sigma} = e^{ikx} \cdot \frac{1}{2} \begin{pmatrix} 1+\sigma \\ 1-\sigma \end{pmatrix} \exp\left(-i\frac{\sigma}{v}\int U_{\sigma,\sigma}dx\right). \quad (2)$$

The wavefunctions (2) contain phase corrections and do not contain backscattering. To account for the backscattering one should include small non-diagonal elements $U_{\sigma,-\sigma}$ ($U_{\sigma,-\sigma} \ll U_{\sigma,\sigma}$) in the consideration. Thus, the diagonal matrix elements of U give inessential corrections to the phases.

Let us consider a single impurity located in point (x_n, y_n) with potential $u(x-x_n, y-y_n)$. The off-diagonal elements of the potential follow from Eq. (10) from [3]. An electron incoming from $-\infty$ in the edge $\sigma = 1$ collides with the impurity and can pass to the edge $\sigma = -1$ with reversion of the momentum $k \to -k$ and the direction of motion. Alternatively, the electron can pass to ∞ conserving k and σ.

To find the amplitude of these processes, one should solve the stationary Schrödinger equation with boundary conditions $\xi = (e^{ikx}, r_n e^{-ikx})$ at $x \to -\infty$ and $\xi = (t_n e^{ikx}, 0)$ at $x \to \infty$, where t_n and r_n are amplitudes of transmission and reflection, accordingly. The unitarity yields $|r_n|^2 + |t_n|^2 = 1$.

In particular, solving the Schrödinger equation in the first Born order, we have

$$r_n = (1/v)e^{2ikx_n}\int_{-\infty}^{\infty} e^{2ikx}u_{1,-1}(x)dx,$$
$$u_{1,-1}(x) = \int dy\psi_1(y)u(x, y-y_n)\psi_{-1}(y). \quad (3)$$

In this approximation $r_n \ll 1$, $t_n \approx 1$. We will assume that the characteristic size of the impurity potential $\rho_0 \ll L$. In this case parameters $r_n = r\exp(2ikx_n)$, where r does not depend on n.

Now, go to the problem of many impurities situated in points (x_n, y_n) (n means the impurity number) with areal density n_s. Here we will not restrict ourselves by the Born case and consider r as a real number between 0 and 1. The wave function between n-th and $(n+1)$-th impurities $(x_n < x < x_{n+1})$ is

$$(a_n e^{ik(x-x_n)}, b_n e^{-ik(x-x_n)}). \quad (4)$$

We assume here that between impurities electrons freely propagate. This requires the absence of impurity potential overlapping: the mean distance between impurities along the x-axis $(n_s L)^{-1}$ is larger than the characteristic size of impurity potential ρ_0. The model under consideration is illustrated in Fig. 1. In points x_n electrons meet the n-th impurity and experience backscatterings. The scattering by the impurities determines the system of algebraic equations for a_n, b_n:

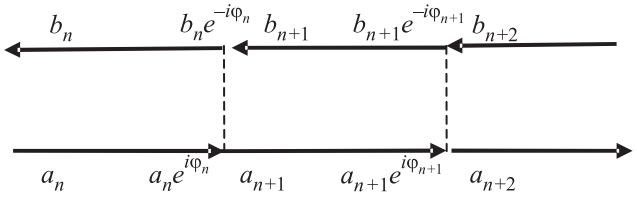

Fig. 1. One-dimensional model of electron localization on the edge states

$$b_n e^{-i\phi_n} = r a_n e^{i\phi_n} + \sqrt{1-r^2}b_{n+1},$$
$$a_{n+1} = \sqrt{1-r^2}a_n e^{i\phi_n} + r b_{n+1}. \quad (5)$$

Here $\phi_n = k(x_{n+1} - x_n)$[2)].

The positions of impurities are random, hence, we can consider ϕ_n as randomly distributed independent numbers. For modeling, we should assume that ϕ_n are randomly distributed within the range $(0, 2\pi)$. This assumption is valid at least if x-distance between subsequent impurities exceeds $2\pi/k$.

In a matrix form Eq. (5) reads

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = S_n \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad (6)$$

where

$$S_n = \frac{1}{\sqrt{1-r^2}} \begin{pmatrix} e^{i\phi_n}(1-2r^2) & re^{-i\phi_n} \\ -re^{i\phi_n} & e^{-i\phi_n} \end{pmatrix}. \quad (7)$$

Consider a finite strip with $N = n_s L\mathcal{L}$ impurities in it. To find the transmission coefficient of the total system we will act like [4]. Namely, we start from $a_1 = 0, b_1 = 1$ ($a_1 = 0$ means no incident wave at $n = 1$, $b_1 = 1$ means normalized to unity intensity of the wave transmitted to the left). Then $|b_N|^{-2}$ gives the transmission coefficient, $G_0|b_N|^{-2}$ is the conductance. The resulting recurrence is

$$\begin{pmatrix} a_N \\ b_N \end{pmatrix} = \left(\prod_{n=1}^{N} S_n\right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (8)$$

In a large system, $\ln|b_N| \propto N$ and the quantity

$$(n_s L)^{-1} \lim_{N\to\infty} N/(2\ln|b_N|)$$

determines the localization length. The calculated inverse localization length (in unites of $(n_s L)$) versus the amplitude of reflection r is represented in Fig. 2. The dependence is approximately quadratic, in accordance

[2)] Note, that the diagonal elements of potential $u_{\sigma,\sigma}$ also can be included in consideration; however, they give corrections to the phases of transmission and reflection which can be accommodated into ϕ_n. The same is valid with respect to the potential of *all* impurities $U_{\sigma,\sigma}$.

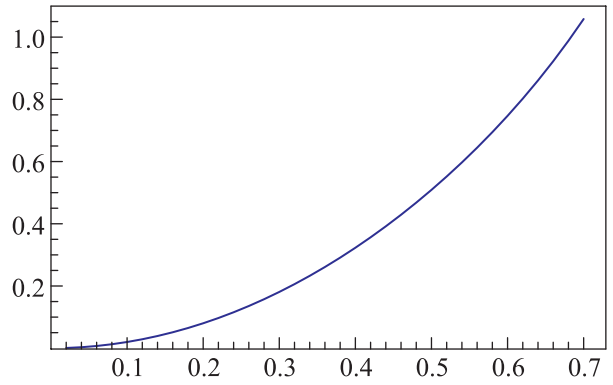

Fig. 2. Inverse localization length (in unites of $n_s L$) *versus* the amplitude of reflection r

with the mean free path calculated in the Born approximation.

Phonon mechanism of electron dephasing. The localization regime considered above requires the coherence of electron wave function. This coherence is violated by interaction of electrons with phonons. A sufficiently strong decoherence revives the kinetic equation applicability. To estimate the transition temperature between these two regimes we need to find the dephasing time and compare it with the interedge backscattering time which determines the kinetics.

Here we are based on the Hamiltonian of electron-phonon interaction presented in [3]. In contrast to [3] we will deal with the forward same-edge phonon-induced scattering $\sigma \to \sigma$, the role of which is to determine the electron dephasing time τ_ϕ.

We will discuss the phonon mechanism in the low-temperature limit when the frequency of emitted/absorbed phonons has the order of electron thermal energy T (in other words, the process is inelastic) and, hence, τ_ϕ coincides with the inelastic forward relaxation time [3)]. Noting, that the main contribution to τ_ϕ arises from the forward scattering, we find

$$\tau_\phi^{-1} = 2\pi \sum_{\pm,\mathbf{q},k'} |c_\mathbf{q}|^2 |J_{q_z}|^2 |J_{q_y;+1,+1}|^2 \delta_{k',k\mp q_x} \times$$
$$\times (N_\mathbf{q} + 1/2 \pm 1/2)[1 - f_0(E_{+1}(k'))]\delta(vk - vk' \mp cq). \quad (9)$$

Here $f_0(E)$ and $N_\mathbf{q}$ are the Fermi and Bose–Einstein distribution functions, other quantities are described in [3]. Using relations $cq \sim T$, $q_x \approx cq/v \ll q$, $q_y \lesssim \lambda_{1,2}$, $q_x \ll q_y < q$, we obtain at $T \to 0$ $J_{q_z} \approx 1$, $J_{q_y;\sigma'\sigma} \approx 1$. Then

$$\tau_\phi^{-1} = \frac{\Lambda^2 T^3}{4\pi\rho c^4 v} \int_0^\infty x^2 dx \frac{\coth(x/2)(e^{2\epsilon} + e^\epsilon)}{(e^{\epsilon - x} + 1)(e^{\epsilon + x} + 1)}, \quad (10)$$

where $\epsilon = (E - E_\mathrm{F})/T$. Expression (10) is valid if $T \ll \lambda_2 c$. The value of τ_ϕ depends on the electron energy. The integral in (10) runs from ≈ 4.2 at $\epsilon = 0$ to $\epsilon^3/3$ at $\epsilon \to \infty$. The value of τ_ϕ averaged with the derivative of the Fermi function $-e^\epsilon/(e^\epsilon + 1)^2$ gives

$$\langle \tau_\phi \rangle = 0.354 \frac{4\pi\rho c^4 v}{\Lambda^2 T^3}. \quad (11)$$

Eq. (11), in fact, gives the mean free time of inelastic forward scattering which is a reasonable estimation for τ_ϕ. One can see that, at a low temperature, τ_ϕ grows $\propto T^{-3}$, but does not contain an exponentially large factor caused by the wave function overlapping.

Let us discuss now the conductivity and conductance of a strip in the presence of phase decoherence. In the infinite sample the kinetic description requires the destruction of coherence between the elastic scattering acts: $\tau_\phi(T) \ll \tau$, where mean free time τ collects all scattering mechanisms. *Vice versa*, the localization occurs if $\tau_\phi(T) \gg \tau$. The transition temperature is determined by a relation $\tau_\phi(T) \sim \tau$, where τ in the low-temperature limit does not depend on T.

Let now the localization condition $\tau_\phi(T) \gg \tau$ be valid, but the sample length $\mathcal{L}$ exceeds dephasing length $l_\phi = v\tau_\phi$. If $\mathcal{L} \gg l_\phi \gg l_\mathrm{loc}$ the sample can be separated by blocks of length l_ϕ with series resistances $\exp(l_\phi/l_\mathrm{loc})/G_0$, so that the conductivity reads $G_0 \exp(-l_\phi/l_\mathrm{loc}) l_\phi/\mathcal{L}$. Hence, unlike the zero temperature case, the conductivity remains finite, but exponentially small.

It should be noted, that the forward-scattering-induced dephasing caused by phonons have been studied recently [5] in the systems with linear spectrum, graphene strips and carbon nanotubes, where backscattering is forbidden. On the contrary, in the 2D TI the backscattering is weakly permitted, but the dephasing helps to establishment of the kinetic equation regime.

Discussion and conclusions. We based on the topological protection of the edge states of an ideal TI strip. In such system the conductance equals to the quantum independently on the strip length. However, we demonstrated that at a low temperature, in the presence of the interedge scattering, a long TI strip, like 1D wire of conventional conductor, experiences the localization of the edge states, and the conductance exponentially tends to zero with increase of the strip length. The localization length becomes exponentially long with respect to the strip width. This understanding resolves the

[3)] Note, that strict consideration of the phase coherence requires the solution of the quantum kinetic equation for the density matrix in σ. Strong dephasing, however, suppresses the off-diagonal elements of this matrix. As a result, the diagonal elements of the density matrix become larger than off-diagonal elements; that revives the classical kinetic equation.

imaginary contradiction between the localization theory and the topological protection.

In addition, we examined the validity limit of localization approach due to the dephasing caused by the phonon-induced intraedge forward scattering. The dephasing leads to the finite conductivity of the system. In the case when the dephasing length is less than the localization length (and also the dephasing length is less than the sample length) the regime of kinetic equation applicability is established. This condition is easy to be fulfilled due to the exponentially large localization length.

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