Entropy spectrum and area spectrum of a modified Schwarzschild black hole via an action invariance

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The spectroscopy of a quantum corrected Schwarzschild black hole in the gravity's rainbow is investigated. By utilizing an action invariance of the black hole and with the help of Bohr–Sommerfield quantization rule, the entropy and area spectrum for the modified black hole are calculated. Here, the quasinormal modes of the black hole is not used. The obtained entropy spectrum is equally spaced and has not dependence on the quantum effects of the spacetime. However, due to the spectime quantum effects of the modified black hole, the obtained area spectra is not equally spaced and the area spacing depend on the horizon area of the black hole. But, as the same to the entropy spectrum, the area spectrum of the gravity's rainbow is independent of the energy of test particle, although the modified spacetime has the dependence on the particle energy.

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The entropy and area spectrum of black holes is a important respect on black hole quantum property. Firstly, by considering a black hole emitting or absorbing a test particle, Bekenstein present an evenly spaced area spectrum with minimum increase of horizon area as $(\triangle A)_{\min} = \varepsilon G\hbar$, in which ε is an undetermined dimensionless constant [1]. Furthermore, Bekenstein prove that, for a slowly evolving black hole, the horizon area is an adiabatic invariant [2–4]. According Ehrenfest principle, any classical adiabatic invariant corresponds to a quantum quantity with discrete spectrum. This way, black hole area is quantized and the area quantum is presented as

$$(\triangle A)_{\min} = 8\pi\hbar. \tag{1}$$

Afterwards, combining the adiabatic invariance of the horizon area and the quasinormal modes (QNMs) of black holes, Kunstatter present an adiabatic invariant formula for black holes [5]. It is showed that, for a system with energy E and vibrational frequency $\omega(E)$, there is a natural adiabatic invariant with $I = \int \frac{dE}{\omega(E)}$. Moreover, as a semiclassical limit, a black hole with transition frequency can be considered as a classical system of periodic motion with the vibrational frequency from the QNMs. Such, following the Bohr–Sommerfeld quantization rule $I = n\hbar$, the entropy spectrum and area spectrum of black holes are produced. It should be pointed that, the method of relating the QNMs to black hole quantization was proposed firstly by Hod [6, 7]. Based on Bohr's correspondence, it is argued that [6, 7], the highly damped QNMs of black holes should be identical to the quantum transition of the corresponding system and the real part of the asymptotic QNMs should equal the quantum transition frequency. Thus, letting the variation of black hole energy equal the radiation energy, the area spectrum of black holes is obtained. The results of black hole quantization from Kunstatter's adiabatic invariant and Hod's method are consistent [5, 6, 8–11]. That is, by these methods, the area spectrum

$$(\triangle A)_{\min} = 4\,(\ln 3)\,G\hbar \tag{2}$$

of black holes is given by using the proper frequency from the real part of the QNMs [5, 6, 9]. Later, a important progress for relating the QNMs to black hole quantization was given. Maggiore suggested that [8], in the semiclassical limit, the transition frequency of black hole should be between quantum levels corresponding the absolute values of the complex QNMs. Moreover, for highly excited black holes, the imaginary part of the QNMs much large than the real part and then the proper frequency is given from the imaginary part. Also, the proper frequency from the imaginary part of the QNMs was pointed out earlier in [12]. In the same way, by using the proper frequency from the imaginary part of the QNMs, the area spectrum (1) is obtained [8, 10, 11]. In the literature, by the adiabatic invariant and the QNMs of black holes, many other investigations have been done for black hole quantization and the area and entropy spectrum in different spacetimes have been given [13–23].

Motivated by the proposal of black hole area is a adiabatic invariant, different adiabatic invariants for obtaining the quantum spectrum of black holes have been

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proposed [24–29]. In [29], by considering the outgoing particles pass through the event horizon, an adiabatic invariant action variable for black holes is proposed and written by Majhi and Vegenas as

$$I = \int p_i dq_i = \int \int_0^{p_i} dp_i' dq_i = \\ = \int \int_0^{p_0} dp_0' dq_0 + \int \int_0^{p_r} dp_r' dq_r, \quad (3)$$

where p_i is the conjugate momentum of the coordinate q_i, $q_0 = \tau$ with τ is the Euclidean time. By applying the Bohr–Sommerfeld quantization rule to the adiabatic invariant, the quantum spectrum for static spherically symmetric non charged black hole is given without using the QNMs of black holes [29]. Then, via showing an action invariance, a method for studying black hole quantization is put forwarded. After that, by using the action invariance of black holes, many studies on black hole spectroscopy have been done and Majhi and Vegenas's work have been extended to different action variables and different spacetime [30–42]. Therein, by considering the canonically invariant of action variable, the adiabatic invariant (3) was rewritten as [30],

$$I = \oint p_i dq_i = \oint \int_0^{p_i} dp_i' dq_i, \quad (4)$$

where q_i is the dynamic degree freedom in Euclidean coordinate, the integral path encircle and very close to the event horizon. Using the modified adiabatic invariant action variable, the equally spaced area and entropy spectrum in different classic spcetime including rotating and charged black holes were obtained and the result of the Bekenstein's original spectrum (1) was presented [30, 34].

In this note, using the action variable formula (4), we extend Majhi and Vegenas's work to a quantum corrected Schwarzschild black hole in the gravity's rainbow [43, 44]. For the modified black hole, the spectroscopy is investigate without using the QNMs. The equally spaced entropy spectrum is obtained and the same result to the literatures [29–42] is presented. However, due to the spacetime quantum effects of the modified black hole, the obtained area spectrum is not equally spaced and the spacing depended the horizon area. This is a different result from the one of the usual black holes by the action variable formula (4) [30, 34].

Let us briefly introduce the modified Schwarzschild black hole from gravity's rainbow. As a deformed special relativity, the double special relativity (DSR) has been proposed [45, 46]. The staring point and the main result of DSR is the modified dispersion relation (MDR) as

$$E^2 f_1^2 (E;\lambda) - p^2 f_2^2 (E;\lambda) = m_0^2, \quad (5)$$

where f_1 and f_2 are two energy functions, λ is a parameter of order the Planck scale. Eq. (5) shows that, MDR is energy dependent. It is to say, particles with different energies have different energy-momentum relations.

In the context of DSR, it is put forwarded that the flat spacetime with Planck scale corrections has the invariant [43, 44]

$$ds^2 = -\frac{dt^2}{f_1^2} + \frac{dr^2}{f_2^2} + \frac{r^2}{f_2^2} d\Omega^2. \quad (6)$$

Then, the DSR spacetime is endowed with an energy dependent metric, namely rainbow metric.

By extending (6) to incorporate curvature, gravity's rainbow has been presented and the modified Schwarzschild solution has been expressed as [44]

$$ds^2 = -\frac{1-2M_t/r}{f_1^2} dt^2 + \frac{1}{f_2^2 (1-2M_t/r)} dr^2 + \frac{r^2}{f_2^2} d\Omega^2. \quad (7)$$

Thus, the present modified Schwarzschild spacetime is endowed with Plank scale modifications behaving as the dependence on the energies of test particles.

Now, using the adiabatic invariant (4), we discus the entropy and area spectrum of the modified Schwarzschild black hole. By the transformation $t \to \to -i\tau$, the Euclidean metric of the black hole (7) can be written as

$$ds^2 = -\frac{1-2iM_\tau/r}{f_1^2} dt^2 + \frac{1}{f_2^2 (1-2iM_\tau/r} dr^2 + \frac{r^2}{f_2^2} d\Omega^2, \quad (8)$$

where M_τ denote the Euclidean mass with [25, 47]

$$M_t \to iM_\tau. \quad (9)$$

For the non charged static spherically symmetric spacetime, the only dynamic degree freedom can be written as q_r and the adiabatic invariant quantity (4) is

$$I = \oint p_r dq_r = \oint \int_0^{p_r} dp_r' dq_r. \quad (10)$$

To proceed with an explicit computation on (10), we remove the momentum p_r by energy and applied the Hamilton's equation

$$\dot{r} = \frac{dr}{d\tau} = \frac{dH_\tau'}{dp_r'}. \quad (11)$$

Here the radial paths $\dot{r}$ is determined by the $t-r$-sector of (8) with $d\Omega^2 = 0$. And, H_τ' is the energy of the black hole after the radiation with energy $\omega = \frac{1}{f_1 f_2}(M - M_\tau')$. It is noted that, the Hamilton H_τ of the modified black hole (8) equal the ADM mass, namely

$$H_\tau = M_{\mathrm{ADM}} = -\frac{1}{8\pi} \int_s \varepsilon_{abcd} \nabla^c \xi^d = \frac{M_\tau}{f_1 f_2}. \quad (12)$$

Then, substituting (11) into (10) yields

$$I = \oint \int_0^{H_\tau} \frac{dH'_\tau}{d\dot{r}} dr = \oint \int_0^{H_\tau} dH'_\tau d\tau. \tag{13}$$

Next, in order to perform the integral on time, the basic characteristics of the event horizon can be used. It has been pointed out, the period of the Euclidean time of a loop about the event horizon is equal 2π multiply the inverse of the surface gravity of black holes [25, 29, 47, 48], namely

$$\oint d\tau = \frac{2\pi}{\kappa_\tau}. \tag{14}$$

Then, by (14), (13) is changed into

$$I = 2\pi \int_0^{H_\tau} \frac{dH'_\tau}{\kappa'_\tau}. \tag{15}$$

Considering the Hawking temperature of the black hole (8) being $T_\tau = \hbar\kappa_\tau/2\pi$, we have

$$I = \int_0^{H_\tau} \frac{dH'_\tau}{T'_\tau}. \tag{16}$$

Thus, for the modified black hole (8), using the first law of black hole thermodynamics

$$\frac{dH'_\tau}{T'_\tau} = dS'_{bh}, \tag{17}$$

the adiabatic invariant (16) can be written as

$$I = \hbar \int_0^{S_{bh}} dS'_{bh} = \hbar S_{bh}. \tag{18}$$

For the present canonical invariant corresponding one dynamic degree freedom q_r, the Bohr–Sommerfeld quantization rule can be used as

$$I = 2\pi n\hbar, \tag{19}$$

where $n = 1, 2, 3, \ldots$. Thus, from (18) and (19), the entropy spectrum of the Schwarzschild modified black hole in gravity's rainbow can be obtained as

$$S_{bh(n)} = \frac{I_n}{\hbar} = 2\pi n, \tag{20}$$

$$\triangle S_{bh} = S_{bh(n+1)} - S_{bh(n)} = 2\pi. \tag{21}$$

It is found that, for the quantum corrected black hole, the equally spaced entropy spectrum is derived. And, the entropy spectrum of the gravity's rainbow is independent of the energies of test particles, although the modified spacetime is energy dependent. Moreover, the presented entropy spectrum is the same result to the entropy spectrum from the other black holes by the action invariance [29–42].

Then, if we use the Bekenstein–Hawking (B–H) formula

$$S_{bh} = \frac{A}{4\hbar}, \tag{22}$$

the area spectrum of the black hole can be obtained from the entropy spectrum formulas (20) and (21) as

$$A_n = 4\hbar S_{bh(n)} = 8\pi n\hbar, \tag{23}$$

$$\triangle A = 4\hbar \triangle S_{bh} = 8\pi\hbar. \tag{24}$$

Obviously, the area spectrum is equally spaced and is the same to the Bekenstein's original result (1). This is the same result as from the usual Schwarzschild black hole [30].

However, due to the Plank scale modifications effects of the MDR, the entropy of the modified black hole in the gravity's rainbow should have quantum corrected items to the B–H entropy [49–53]. Among them, a corrected entropy formula can be written as [52, 53]

$$S = \frac{A}{4\hbar} + \frac{1}{2}\ln\left(\frac{A}{4\hbar} + \frac{1}{2}\right) + c, \tag{25}$$

where c is a constant. Thus, based on the entropy (25) and the entropy spectrum (20) and (21), the area spectrum of the modified black hole should be different from (23) and (24). Let us giving a discuss. By a variation on (25), we obtain that

$$\triangle S = \frac{dS}{dA}\triangle A = \left(\frac{1}{4\hbar} + \frac{1}{2A + \hbar}\right)\triangle A = \frac{\triangle A}{4\hbar}\frac{A + 4\hbar}{A + 2\hbar} \tag{26}$$

and

$$\triangle A = \frac{A + 2\hbar}{A + 4\hbar} \cdot 4\hbar \triangle S. \tag{27}$$

Then, from (21), the area spacing of the modified black hole is obtained as

$$\triangle A = 8\pi\hbar \frac{A + 2\hbar}{A + 4\hbar} = 8\pi\hbar\left(1 - \frac{2\hbar}{A + 4\hbar}\right). \tag{28}$$

It is find that, considering the spacetime quantum effects of black holes, the area spectrum of the modified Schwarzschild black holes is not equally spaced and the area spacing depended on the horizon area. This is a different result to the result by the action variable from the usual Schwarzschild black hole [29, 30]. But, as the same as the entropy spectrum, the obtained area spacing is independent of the particle's energy. And, by ignoring the quantum corrected item of the entropy (25), the black hole entropy naturally came back to the B–H entropy and the area spectrum (28) can return to the equally spaced area spectrum (24).

In summary, by showing the adiabatic invariant action variable (3), a method for studying black hole quantization is put forwarded by Majhi and Vegenas [29]. Next, by considering the canonically invariant of action variable, the action variable is modified as (4) [30]. In this note, by using the modified action variable (4), Majhi and Vegenas's work was extended to a quantum corrected spacetime in the gravity's rainbow and the spectroscopy of the modified Schwarzschild black hole is investigated. The entropy spectrum and area spectrum of the modified black hole were calculated by using the action variable and with the help of Bohr–Sommerfield quantization rule. Here, the QNMs of black holes was not used. The equally spaced entropy spectrum was derived and it is consistent to the results in not only the normal Schwarzschild black hole but also other static and rotating black holes by the action invariance [29–42]. And that, the entropy spectrum is independent the energy of test particle, although the modified spacetime is energy dependent. That is, the entropy spectrum of the modified Schwarzschild black hole in the gravity's rainbow has not dependence on the Plank scale modifications of the MDR. In this sense, in supporting the universality of black hole entropy spectrum, the note provided an illustration via the action invariance of black holes. However, due to the spectime quantum effects of the modified black hole, the obtained area spectra is not equally spaced and the area spacing depended on the horizon area. This is a different result from the one of the usual Schwarzschild black hole. But, when the quantum effect of the modified black hole is ignored, the area spectrum of the black hole can come back to the case of the usual Schwarzschild black hole and the same equally spaced spectrum as the Bekenstein original result (1) can be obtained. In addition, it was found that, as the same to the entropy spectrum, the area spectrum of the gravity's rainbow is not energy dependent.

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