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# Occurrence of flat bands in strongly correlated Fermi systems and high-$T_c$ superconductivity of electron-doped compounds

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We consider a class of strongly correlated Fermi systems that exhibit an interaction-induced flat band pinned to the Fermi surface, and generalize the Landau strategy to accommodate a flat band and apply the more comprehensive theory to electron systems of solids. The non-Fermi-liquid behavior that emerges is compared with relevant experimental data on heavy-fermion metals and electron-doped high-$T_c$ compounds. We elucidate how heavy-fermion metals have extremely low superconducting transition temperature $T_c$, its maximum reached in the heavy-fermion metal $CeCoIn_5$ does not exceed 2.3 K, and explain the enhancement of $T_c$ observed in high-$T_c$ superconductors. We show that the coefficient $A_1$ of the $T$-linear resistivity scales with $T_c$, in agreement with the experimental behavior uncovered in the electron-doped materials. We have also constructed schematic temperature-doping phase diagram of the copper oxide superconductor $La_{2-x}Ce_xCuO_4$ and explained the doping dependence of its resistivity.

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An especially challenging task confronting present-day condensed matter theory is explication of the elusive origin of the non-Fermi-liquid (NFL) behavior observed in strongly correlated Fermi systems beyond a critical point where the low-temperature density of states $N(T \to 0)$ diverges without breaking any symmetry inherent of the ground state. In homogeneous matter, the case to be addressed in this article, such a divergence is associated with the onset of a *topological* transition (TT) signaled by the emergence of an additional root $p = p_b$ of the condition [1, 2]

$$\epsilon(p, \lambda_t; T = 0) = 0 \tag{1}$$

for vanishing of the single-particle energy measured from the chemical potential $\mu$, otherwise satisfied only by $p = p_F$, the Fermi momentum. Thus, if $\lambda_t$ is the critical coupling constant for onset of the posited TT, the curve $\epsilon(p, \lambda_t)$ must touch the axis $p$ at the bifurcation momentum $p = p_b$. Accordingly, at $p \to p_b$ the group velocity $v(p) = \partial\epsilon(p, \lambda_t)/\partial p$ vanishes as $\sqrt{\epsilon(p)}$ to yield

$$N(T \to 0) \propto T^{-1/2}. \tag{2}$$

In an original scenario advanced by I. M. Lifshitz more than fifty years ago [1] and recently applied to describe electron systems of heavy-fermion metals [3, 4], some TTs are assumed to occur in systems of *noninteracting* electrons moving in the external field of the crystal lattice. In such a case, the quasiparticle occupation numbers adhere to the discrete Fermi-liquid (FL) values $n(p) = 0, 1$ beyond the Lifshitz topological transition (LTT) point, while the Fermi surface becomes multi-connected. Topological transitions may occur in systems of *interacting* fermions as well [5]. Within the FL approach, the feasibility of the Lifshitz ("bubble" or "pocket") scenario for the rearrangement of the Landau state was considered [6, 7] within the framework of the Landau equation [8]

$$\frac{\partial\epsilon(p,T)}{\partial\mathbf{p}} = \frac{\mathbf{p}}{M} + \int f(\mathbf{p},\mathbf{p}')\frac{\partial n(p',T)}{\partial\mathbf{p}'}dv' \tag{3}$$

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for the single-particle spectrum $\epsilon(p)$. Here $M$ is the mass of a free fermion, $d\upsilon = 2d^3p/(2\pi)^3$ is the three-dimensional volume element,

$$n(p,T) = (1 + e^{\epsilon(p)/T})^{-1} \tag{4}$$

is quasiparticle momentum distribution, and $f(\mathbf{p},\mathbf{p}')$ is a phenomenological interaction function. Eq. (3) was derived by Landau [8] from the equality

$$\int \mathbf{p}\, n(p) d\upsilon = M \int \frac{\partial \epsilon(p)}{\partial \mathbf{p}} n(p) d\upsilon \tag{5}$$

between the momentum of the system, moving with velocity $\delta\mathbf{v}$, its mass flow stemming from Galilean invariance. We note that Eq. (5) is also valid when the model of heavy-fermion liquid is applicable, as it is in our present case [9, 10]. Equation (3) is then obtained upon retaining only the leading terms in $\delta n(p)$ on both sides of Eq. (5) while invoking the FL relation

$$\delta\epsilon(\mathbf{p},n) = \sum_{\mathbf{p}'} f(\mathbf{p},\mathbf{p}')\delta n(p'). \tag{6}$$

As will be seen, Eq. (5) allows one to introduce and explore a different, *interaction-induced* type of rearrangement of the Landau state, often called fermion condensation and described more vividly as a swelling of the Fermi surface [11, 13, 12]. This phenomenon has its genesis in a proliferation of the number of roots of Eq. (1) to form a *continuum*. (For recent articles on this topic, see Refs. [9, 10, 14–20].)

The emergence of such a phase transition in homogeneous matter can be elucidated within the original Landau approach to FL theory, in which the ground-state energy $E$ is postulated to be a functional of the quasiparticle momentum distribution $n(p)$. At $T = 0$, the onset of fermion condensation in homogeneous matter is attributed to the occurrence of a nontrivial solution $n_*(p)$ of the variational condition [11]

$$\frac{\delta E(n,\lambda)}{\delta n(p)} - \mu = 0, \quad p \in [p_l, p_u], \tag{7}$$

in a finite momentum interval surrounding $p_b$, the chemical potential $\mu$ being determined from the Landau postulate that the quasiparticle and particle numbers *coincide*. It is Eq. (7) that describes a flat band pinned to the Fermi surface, here also referred to as the fermion condensate (FC). Outside the FC domain $[p_l, p_u]$ the usual FL occupancies still apply: $n_*(p) = 1$ at $p < p_l$ and $n_*(p) = 0$ at $p > p_u$. However, the occupation numbers inside the FC, evaluated through Eq. (7), change *continuously* between 1 and 0 with increasing $p$. The volume occupied in momentum space by quasiparticles with nonzero probability is no longer just $p_{\rm F}^3/3\pi^2$ but is instead given by the relation [15]

$$\int_0^{p_l} d\upsilon + \int_{p_l}^{p_u} n_*(p) d\upsilon = \frac{p_{\rm F}^3}{3\pi^2}. \tag{8}$$

We now turn to the demonstration that Landau equation (3) is to be modified in dealing with Fermi systems harboring a FC. It is revealed by examination of analytic properties of solutions of this equation in systems where the interaction function $f(\mathbf{p},\mathbf{p}')$ has no singularities in momentum space. For this case, it has been established [7] that the solutions of Eq. (3) are in fact *analytic* functions of momentum $p$ in the full momentum space. However, this property is lost if the system hosts a FC. Indeed, the left side of Eq. (7) is nothing but the quasiparticle energy measured from the Fermi surface, which vanishes identically in the FC domain. Evidently, the FC domain cannot occupy the full momentum space – a fact confirmed in analytically soluble models of fermion condensation. Thereby we arrive at the strong conclusion that the single-particle spectrum $\epsilon(p, n_*)$ of the problem with FC present must be a *non-analytic* function of momentum $p$, for if an any analytic function vanishes identically in some domain, it must vanish everywhere. This implies that in the case where the interaction function $f$ has no singularities in momentum space, the FC solutions $\epsilon(p, n_*)$ *cannot meet* Eq. (3), and therefore in systems with a FC, Landau equation (3) must be modified to allow for such an eventuality. This modification can be made on the basis of Eq. (5) along the same lines as Eq. (3) was obtained in Landau theory. The profound difference is that the system harboring a FC is, in fact, a *two-component* system, as made quite evident in Eq. (8). Correspondingly, variations of the quasiparticle momentum distribution $n(p)$ lead to the closed equation

$$0 = \frac{\mathbf{p}}{M} + \int f(\mathbf{p},\mathbf{p}') \frac{\partial n_*(p')}{\partial \mathbf{p}'} d\upsilon', \quad p, p' \in [p_l, p_u] \tag{9}$$

for determining $n(p) = n_*(p)$ inside the FC domain derived with accounting for the fact that outside the FC region, $dn_*(p)/dp = 0$. In arriving at this relation, we have ensured consistency with the central feature of the FC-inhabited state that the group velocity *vanishes identically* in the interval $[p_l, p_u]$, whose range is found with the aid of the requirement $n_*(p) < 1$. The quasiparticle spectrum of the *normal* component obeys the Landau-type equation

$$\frac{\partial \epsilon(p)}{\partial \mathbf{p}} = \frac{\mathbf{p}}{M} + \int f(\mathbf{p},\mathbf{p}') \frac{\partial n_*(p')}{\partial \mathbf{p}'} d\upsilon', \quad p \notin [p_l, p_u], \tag{10}$$

and in effect $p' \in [p_l, p_u]$, derived in the same way as Eq. (3) from Eq. (5). Equation (9) can also be derived by differentiation of the basic equation (7) with respect to momentum $\mathbf{p}$, then adopting the relation (6) between interaction-induced variations of relevant quantities, rewritten as $\delta\epsilon(p) = \delta\epsilon_0 + f\delta n(p)$, where $\delta\epsilon_0$ is the variation of the spectrum of noninteracting quasiparticles. In homogeneous matter, where $\partial\epsilon_0/\partial\mathbf{p}$ is simply $\mathbf{p}/M$, we are led to Eq. (9). The special convenience of this route lies in the opportunity to generalize Eq. (9) for analysis of the FC phenomenon in electron systems of solids. In doing so, one needs to replace $\mathbf{p}/M$ by the corresponding derivative $\partial\epsilon_0/\partial\mathbf{p}$, evaluated, say, within the tight-binding model, or within a more advanced microscopic description of the electron spectrum $\epsilon_0(\mathbf{p})$. A definite mathematical signature identifies those interacting many-fermion systems whose single-particle spectrum exhibits a flat portion. This is the topological charge (TC) of the corresponding ground state. For a system containing a FC, the TC must take a *half-odd-integral* value, whereas the TC of any unorthodox state featuring one or more Lifshitz pockets is always *integral* [12].

It is instructive to compare changes that occur in the fundamental equation (3) beyond the point of fermion condensation with those that occur at a second-order phase transition. In the latter case, some symmetry inherent in the Landau ground state is broken, and a corresponding order parameter comes into play (notably, the gap $\Delta(p)$ in superfluid FL's), dramatically rearranging Eq. (3) and both the key FL quantities, i.e., the momentum distribution $n_{\rm FL}(p) = \theta(p_{\rm F} - p)$ and the single-particle spectrum $\epsilon_{\rm FL}(p) = p_{\rm F}(p - p_{\rm F})/M^*$, where $M^*$ is the effective mass. In the FC topological phase transition no such order parameter enters; nevertheless Eq. (3) alters, being replaced by the set of *two equations* (9) and (10), in harmony with the two-component character of the phase containing a FC. Although the equations written above to describe a system containing a FC were derived for zero temperature, they do not strictly apply there, because states involving a FC have the distinctive property of carrying *nonzero residual entropy*

$$S_0 = -\sum \{n_*(p)\ln n_*(p) + [1 - n_*(p)]\ln(1 - n_*(p))\}. \tag{11}$$

Such behavior would contradict the Nernst theorem $S(T = 0) = 0$, if it continued to $T = 0$. Resolution of the incipient contradiction must lie in release of the excess entropy $S_0$ via one or another phase transition associated, especially, with pairing or magnetic ordering and occurring at a respective transition temperature $T_c$ or $T_N$ as $T \to 0$. Even so, the above equations do yield an approximate description, in the same sense as Landau FL theory furnishes a viable approximate theory of the low-temperature properties of liquid $^3$He *despite* the fact that this system undergoes a normal-to-superfluid transition as $T \to 0$, simply because of its extremely low critical temperature $T_c$ for termination of superfluidity. By the same token, as long as the corresponding $T_c$ is low enough, Eqs. (9) and (10) support an acceptable zeroth approximation for a system of fermions having a FC.

To properly address the finite-temperature case, the energy $E$ must be replaced by the free energy $F$. So modified, the variational condition (7) takes the Landau-like form (cf. Eq. (4))

$$\epsilon(p, n_*) = T\ln\frac{1 - n_*(p,T)}{n_*(p,T)}. \tag{12}$$

Taking the solution $n_*(p)$ of Eq. (7) or (9) as a zeroth approximation for $n_*(p,T)$ applicable at low $T$, we infer that at finite temperatures the dispersion of the spectrum in the momentum interval $[p_l, p_u]$ becomes proportional to $T$ [13]. (Since the change of the NFL occupation numbers $n_*(p)$ incurred by finite $T$ in this region remains quite small, we shall continue to use the term FC for the totality of the corresponding single-particle states.) The temperature-dependent generalization of the above FC equations is obtained by inserting the derivative

$$\mathbf{v}(\mathbf{p}) \equiv \frac{\partial\epsilon(p,T)}{\partial\mathbf{p}} = -T\frac{\partial n_*(p,T)/\partial\mathbf{p}}{n_*(p,T)[1 - n_*(p,T)]} \tag{13}$$

into the left side of Eq. (9) to yield, in the FC domain,

$$-T\frac{\partial n_*(p,T)/\partial\mathbf{p}}{n_*(p,T)[1-n_*(p,T)]} = \frac{\mathbf{p}}{M} + \int f(\mathbf{p},\mathbf{p}')\frac{\partial n_*(p',T)}{\partial\mathbf{p}'}dv'. \tag{14}$$

This equation holds in the FC interval $p \in [p_l, p_u]$, whose length shrinks with $T$, vanishing at an upper critical temperature $T_f$. The presence of the FC phase on the disordered side of the transition, where standard FL theory should ordinarily exert dominion, results in the breakdown of virtually all thermodynamic and kinetic predictions of Landau FL theory. Further discussion will be focused on the magnitude and temperature dependence of the low-$T$ resistivity in the disordered, FC phase, which may represent the most conspicuous departures from FL theory. In *elastic* scattering processes involving normal quasiparticles and the FC, the condensate behaves as a system of *impurities*, giving rise to a residual resistivity $\rho_0$ in *clean* metals, whose value *depends* on the pressure $P$. Experimental documentation of the consequent impact on $\rho_0$ (an effect inconceivable within the textbook understanding of kinetic

phenomena in Fermi liquids) requires high-quality samples without doping $x$ that introduces substantial disorder. In spite of this stricture, a profound effect has been observed in measurements of the resistivity of the heavy-fermion compounds $CeCoIn_5$ and $CeAgSb_2$ [21, 22]. The latter compound undergoes an antiferromagnetic transition, whose critical temperature declines under pressure. On the ordered side of this transition, $\rho(T)$ demonstrates the usual FL behavior $\rho(T) = \rho_0 + A_2T^2$, with residual resistivity $\rho_0$ below 0.5 $\mu\Omega\,$cm. However, with increasing $P$ this familiar behavior of $\rho(T)$ is disrupted. As $P$ approaches the value $P_c$ at which magnetic ordering is destroyed, the residual resistivity $\rho_0$ abruptly soars upward, reaching values of about 50 $\mu\Omega\,$cm on the disordered side of the transition [22]. We submit that such a jump of $\rho_0$ amounts to irrefutable evidence for the presence of a FC on the disordered side of the phase transition.

The heavy-fermion metal $CeCoIn_5$ proves to be a superconductor on the ordered side of the corresponding phase transition. We contend that in such systems, NFL behavior of $\rho(T)$ on the disordered side of this transition is primarily associated with *inelastic* scattering processes in which FC quasiparticles are generated or eliminated *singly* so as to augment the FL formula for $\rho(T)$ with a term linear in $T$, whose magnitude $A_1$ is proportional to the FC range $L_f$ [23–25]:

$$A_1(x) \propto L_f(x). \tag{15}$$

The low-$T$ resistivity $\rho(T, P = 0)$ found experimentally for the normal state of $CeCoIn_5$ is in fact consistent with this behavior [26]. When the pressure $P$ is raised to a critical value $P^* \simeq 1.6\,$GPa, it is found that in a narrow pressure diapason around $P^*$, $CeCoIn_5$ exhibits a crossover to Landau-like behavior $\rho(T) = \rho_0 + A_2T^2$. Concomitantly, the residual resistivity $\rho_0$ is reduced tenfold in a topological transition from a state with a FC to one without, dropping to a very small value around 0.2 $\mu\Omega\,$cm that is due solely to impurity scattering. The crucial difference between the effects of the FC and the impurities that are present lies in the fact that the former – belonging as it does to the system of electrons – contributes to the resistivity *entirely by virtue of Umklapp processes* occurring in the crystal lattice, which spoils electron momentum conservation. However, these processes play no significant role in the thermal conductivity. One is led to the conclusion that the Lorenz number

$$L_0 = \lim_{T\to 0} \kappa(T)/[T\sigma(T)], \tag{16}$$

evaluated at low pressures $P < P^*$ where the FC is present, must be suppressed compared with the textbook value $L_0 = \pi^2/3e^2$. It is just such a violation of the Wiedemann–Franz law that notoriously occurs in $CeCoIn_5$ [26].

These new insights into the observed NFL behavior of $\rho(T)$ may be elaborated as follows. Replacement of a normal particle in a scattering diagram by a FC quasiparticle entails replacement, in the formula for the resistivity $\rho(T)$, of the $T$-independent density of states $N_{\rm FL}$ of Fermi-liquid theory by the FC density of states $N(T) \propto T^{-1}$, thereby promoting linear dependence of $\rho(T)$ on $T$. On the other hand, at $T > T_f$ where the effect of the LTT point still persists, the quantity $N_{\rm FL}$ is replaced by $N(T) \propto T^{-1/2}$ in accordance with Eq. (2), yielding $\rho(T) \propto T^{3/2}$. Such behavior closely resembles the NFL behavior $\rho(T) \propto T^{1.5\pm0.1}$ revealed in measurements of the resistivity of the normal state of $CeCoIn_5$ as well as in *electron-doped* high-$T_c$ superconductors [27, 28]. Moreover, the experimental $T-P$ phase diagram of $CeCoIn_5$ shown in Fig. 4 of Ref. [21] and the experimental $T-x$ phase diagram of $La_{2-x}Ce_xCuO_4$ belonging to the LCCO family presented in Fig. 1 of Ref. [28] are very much alike. This empirical fact casts serious doubt on the premise that the Kondo effect bears sole responsibility for the NFL behavior of the resistivity $\rho(T)$ of heavy-fermion metals, since the Kondo effect is absent in electron-doped compounds. In both of these compounds, release of the residual entropy $S_0$ occurs through the antiferromagnetic phase transition that can be replaced by the superconducting phase transition at lowering temperatures, as it is seen from Fig. 1.

Let us now discuss the enhancement of the superconducting critical temperature $T_c$, attributed to critical spin-fluctuations in Refs. [29, 28]. Within the flat-band scenario, this enhancement is estimated on the base of the standard BCS equation

$$1 = \mathcal{V} \int \tanh\left[\epsilon(p, T_c)/2T_c\right] /2\epsilon(p, T_c) d\upsilon. \tag{17}$$

(For simplicity, we ignore the momentum dependence of the pairing interaction, replacing it by the constant $\mathcal{V}$.) Upon manipulations based on Eq. (12) the BCS Eq. (17) takes the form

$$1 = -0.5\mathcal{V}[\alpha\eta n/T_c + N_n(0) \ln(\Omega_D/T_c)] \tag{18}$$

with total electron density $n$, the dimensionless FC parameter $\eta = (p_u - p_l)/p_{\rm F}$ and numerical factor $\alpha = O(1)$. In obtaining this result we retain merely leading terms that diverge at $T_c \to 0$. The first term in square brackets comes from momentum integration over the FC region. The second, usual BCS one is associated with contributions from bands where normal quasiparticles reside. It contains the familiar density of states

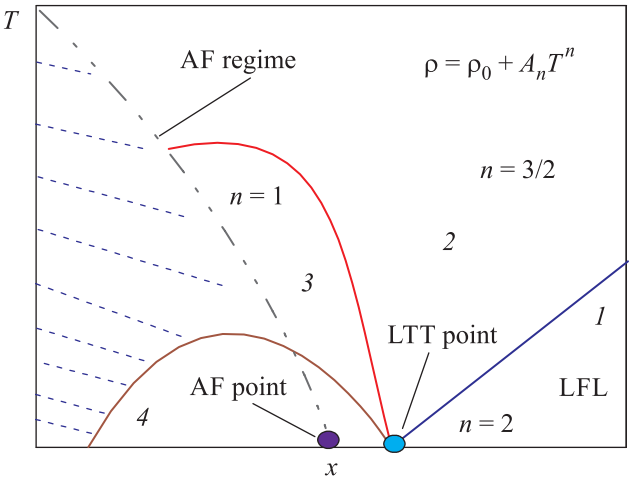


Fig. 1. Schematic temperature-doping $T-x$ phase diagram of $La_{2-x}Ce_xCuO_4$. The exponents $n$ of the resistivity $\rho = \rho_0 + A_nT^n$ are shown for three different regimes, that appear around the LTT point, labeled *1*, *2*, *3*. The Landau–Fermi liquid regime with $\rho(T) \propto A_2T^2$ dependence is labeled by LFL. The transition areas between the corresponding regimes are indicated by solid curves. Both the LTT point at $x = x_{\rm LTT}$ and the antiferromagnetic point under the superconductivity dome (labeled *4*) are indicated by arrows. The onset of antiferromagnetism is traced by the dash-dot-dot line

$N_n(0) \propto p_{\rm F}M_n^*/\pi^2$. In the first approximation, the BCS term can be neglected, and then from Eq. (18) one finds

$$T_c(x) \propto \eta(x), \tag{19}$$

i.e. critical temperature $T_c$ turns out to be linear function of the FC parameter $\eta$. Comparing Eqs. (19) and (15), we infer that both the $A_1$ term in the resistivity $\rho(T)$ and critical temperature $T_c$ change with input parameters $P$ and $\eta$ proportionally to the FC parameter $\eta$. Thus, the theoretical ratio $T_c/A_1$ is approximately independent of the input, in agreement with the experimental behavior uncovered in the electron-doped materials LCCO and PCCO [28, 29].

We note in passing that the departure from the bare mass $M$ of the effective mass $M_n^*$, extracted from experimental data on the specific heat, might be significant, as e.g. in the heavy-fermion metals. This deviation plays important role in the magnitude of $T_c$. To demonstrate let us slightly facilitate Eq. (18) by excluding the effective coupling constant $\mathcal{V}$ with the replacement of the unity on the l.h.s. of Eq. (18) from BCS relation $1 = -0.5\mathcal{V}N_n(0)\ln\left(\Omega_D/T_c^{\rm BCS}\right)$ that yields

$$\frac{T_c}{\epsilon_{\rm F}^0}\ln\left(T_c/T_c^{\rm BCS}\right) = \alpha\eta\frac{M}{M_n^*} \tag{20}$$

with $\epsilon_{\rm F}^0 = p_{\rm F}^2/2M$. The BCS case $T_c = T_c^{\rm BCS}$ is realized provided the FC density $\eta \to 0$. Curiously, the BCS situation also takes place in the heavy fermion metals, even with the flat bands, because of the smallness of the ratio $M/M_n^* \simeq 10^{-2}$–$10^{-3}$. This explains how heavy-fermion metals have extremely low $T_c$, its maximum reached in the heavy-fermion metal $CeCoIn_5$ does not exceed 2.3 K. In dealing with other electron systems where the ratio $M/M_n^*$ is not as small as in the heavy-fermion metals the situation changes. Assuming the needed attraction to come from the electron-phonon exchange and choosing $M/M_n^* = 1$, $\alpha = 0.1$, $\eta = 0.1$, one finds

$$T_c \simeq 0.3\cdot 10^{-2}\epsilon_{\rm F}^0 \tag{21}$$

that explains the enhancement of $T_c$ observed in high-$T_c$ compounds and, particularly, in electron-doped high-$T_c$ materials. From Eq. (21) we infer that the superconducting phase transition in 2D electron gas of MOSFETs and some heterostructures where the needed attraction in the Cooper channel is furnished by the electron-phonon exchange may have sufficiently high $T_c$ provided the FC onset occurs near a critical density, at which the electron effective mass diverges [30]. Noteworthy, in the heavy-fermion metal $CeCoIn_5$, the BCS and FC contributions to $T_c$ are rendered comparable by virtue of the characteristic effective-mass enhancement. This entails, in turn, a marked extension of the boundary between SC and Landau Fermi liquid (LFL) regimes in the $T-P$ phase diagram.

It is a remarkable feature of models of fermion condensation [11, 13, 31] that nontrivial solutions of Eq. (7) carrying a residual entropy $S_0$ often emerge just beyond the LTT point where the resistivity $\rho(T)$ varies as $T^{3/2}$. As a result, near the LTT point the two NFL regimes, proportional respectively to $T^{3/2}$ and $T$, are situated adjacent to each other, as it is shown in the schematic phase diagram 1. It is seen from Fig. 1, that four different regimes for the resistivity behavior come into play in the immediate vicinity of FC onset: the LFL regime $\rho(T) \propto T^2$ (*1*), the LTT-point regime $\rho(T) \propto T^{3/2}$ (*2*), the FC regime $\rho(T) \propto T$ (*3*), and the high-$T_c$ superconducting regime with $\rho = 0$ (*4*). To confirm our analysis of the phase diagram shown in Fig. 1, we consider the behavior of the resistivity in the regions *1* and *3*. In the region *3*, as it is seen from Fig. 2 and follows from the above discussion, the coefficient $A_1 \to 0$ as the doping $x$ approaches the LTT point from above, while in accordance with Eq. (20) $T_c \to 0$. On the other hand, as the doping $x$ tends to the LTT point from below, the coefficient $A_2$ diverges, for in the LFL regime due to

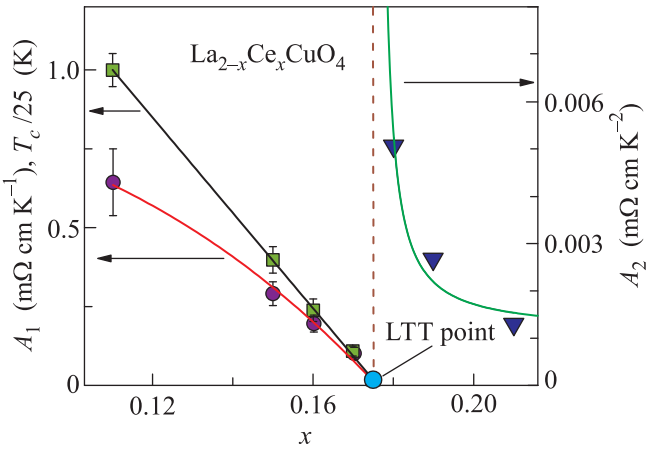


Fig. 2. Doping dependence of $T_c(x)$, $A_1(x)$, and $A_2(x)$ in zero field [28]. The left vertical axis shows the coefficient $A_1$ and $T_c$ divided by 25, while the right vertical axis shows the coefficient $A_2$. In the LFL regime, as the system approaches the LTT point, shown by the arrow, and the doping $x \to x_{\rm LTT}$ from higher doping, the coefficient $A_2$ increases in accordance with Eq. (22), as it is depicted by the solid curve

the conservation of the Kadowaki–Woods ratio one has $A_2 \propto (M^*)^2$ [32, 33]. As a result, one has [9, 10]

$$(M^*(x))^2 \propto A_2 \simeq \left(a_1 + \frac{a_2}{x - x_{LTT}}\right)^2, \qquad (22)$$

where $a_1$ and $a_2$ are constants and $x_{LTT}$ is the doping $x$ at which the LTT point takes place. It seen from Fig. 2, the right panel, that the behavior of $(M^*)^2$ given by Eq. (22) is in agreement with the experimental facts.

In conclusion, prominent on the scene of this seminal subfield of condensed matter physics are numerous versions of Hertz–Millis–Moriya (HMM) theory that ascribe NFL behavior of strongly correlated Fermi systems to quantum critical fluctuations. In stark contrast to these theories, experimental studies of the class of such systems addressed herein furnish clear evidence of the persistence of NFL behavior in regions far enough from the lines $T_c$ and/or $T_{\rm N}(x)$ that the influence of critical fluctuations should be minimal. The crux of the matter is that the single-particle degrees of freedom are the real playmakers for NFL behavior, and the traditional mission of integrating them out is futile, as in throwing the baby out with the bath water.

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