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Classical integrable systems and Knizhnik–Zamolodchikov–Bernard equations

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This paper is a short review of results obtained as part of The Russian Foundation for Basic Research project 12-02-00594. We mainly focus on interrelations between classical integrable systems, Painlevé–Schlesinger equations and related algebraic structures such as classical and quantum R-matrices. The constructions are explained in terms of simplest examples.

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1. Zero curvature equations. We consider integrable equations in classical and quantum Hamiltonian mechanics. In classical mechanics they are described usually by the Lax equations

$$\partial_t L = [L, M]. \tag{1}$$

Here L and M are matrices (operators), depending on the phase variables $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$, S and additional (spectral) parameter z, $L = L(u, v, S; z)$, $M = M(u, v, S; z)$. We assume that $z \in \Sigma$ where Σ is a torus or its degenerations. The variables u, v are the canonical Darboux variable $\{v_j, u_k\} = \delta_{jk}$. The Lax equations can be derived from the $d = 4$ (super) Yang–Mills theories with the gauge group G compactified on Σ. In this case L is identified with a scalar field taking values in the adjoint representations (the Higgs field) restricted on Σ, while M is an element of the Lie algebra of the gauge transformations. The variables S are elements of the Lie algebra $\mathrm{Lie}(G)$. They Poisson commute with (u, v) and their brackets are the Poisson–Lie brackets on $\mathrm{Lie}(G)$. In terms of the Lax operators the Poisson brackets are defined by means of the classical r-matrices (see examples below).

Eq. (1) describes an autonomous Hamiltonian integrable mechanics. To come to the non-autonomous Hamiltonian system we replace (1) by

$$\partial_t L - \kappa \partial_z M = [L, M], \tag{2}$$

where κ is a parameter. This equation is the monodromy preserving condition for the linear equation

$$(\kappa \partial_z + L)\psi = 0, \tag{3}$$

and L now plays the role of connection. In particular, (2) describes the Painlevé equation, Schlesinger system and their generalizations. In the limit $\kappa \to 0$ we come to (1).

Another generalizations of (1) are the Zakharov–Shabat equations for 1+1 integrable field theories which possess the soliton type solutions:

$$\partial_t L - k \partial_x M = [L, M]. \tag{4}$$

One can also consider a generalization of (4) and (2) given by

$$\partial_t L - \kappa \partial_z M - k \partial_x M = [L, M]. \tag{5}$$

We refer to the models described by this equation as the Painlevé field theories.

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The zero curvature equations (1)–(4) keep their forms under the gauge transformations

$$\begin{aligned} D+L &\longrightarrow D+gLg^{-1}-(Dg)g^{-1},\\ \partial_t+M &\longrightarrow \partial_t+gMg^{-1}-(\partial_t g)g^{-1}, \end{aligned} \tag{6}$$

where the differential operator D is given by

$$D=0,\quad \Delta=\kappa\partial_z,\quad D=k\partial_x,\quad D=k\partial_x+\kappa\partial_z \tag{7}$$

for the Eq. (1), (2), (4), (5) respectively.

The purpose of the paper is to show that all types of zero-curvature equations can be described in a similar way, i.e. there exist a universal type Lax pairs which can be used for all the cases. They describe a wide class of integrable systems and related problems. We start from the classical integrable mechanics, which deals with two types of models – many-body systems (interacting particles) and integrable cases of motion of solid body in multidimensional space. First, we demonstrate that many-body systems of Calogero–Ruijsenaars type can be formulated as integrable tops of Euler–Arnold type. Using special gauge transformation the Ruijsenaars–Schneider model is represented in the form

$$L(z,S,\eta)=\mathrm{tr}_2[R^{\eta}_{12}(z)S_2], \tag{8}$$

where the relativistic deformation parameter η enters the Lax matrix as the Planck constant of a certain quantum R-matrix. Being formulated as tops the many-body systems are then naturally included into a more general class of integrable models, which consists of spin chains and Gaudin models. The top-like description also makes it easy to pass to 1+1 integrable equations including one-dimensional Landau–Lifshitz type magnetics, principal chiral models and their generalizations. At the same time the Gaudin models can be considered as autonomous version of the Schlesinger systems – the monodromy preserving equations which can be reduced to Painlevé equations. Finally, we come to the quantum version of the Schlesinger models described by the Knizhnik–Zamolodchikov–Bernard (KZB) equations well-known in studies of conformal field theories. The consistency condition for the KZB connections is guarantied by identities for the initial R-matrix entering (8). In the end we briefly discuss that the equations of Painlevé–Schlesinger type can be generalized to the so-called Painlevé field theories.

2. Calogero-Moser model as integrable top. Let us start with the most simple example (more complicated and general cases can be found in [1–3]) – 2-body Calogero–Moser model. The Hamiltonian is given as

$$H^{\mathrm{CM}}=\frac{1}{2}v^2-\nu\frac{v}{2u}=\frac{1}{2}\left(v-\frac{\nu}{2u}\right)^2-\frac{1}{2}\frac{\nu^2}{(2u)^2} \tag{9}$$

in the canonical coordinates $\{v,u\}=1$. Its Lax matrix

$$L^{\mathrm{CM}}(z)=\begin{pmatrix} v-\frac{\nu}{2u}+\frac{\nu}{z} & \frac{\nu}{2u}+\frac{\nu}{z}\\ -\frac{\nu}{2u}+\frac{\nu}{z} & -v+\frac{\nu}{2u}+\frac{\nu}{z}\end{pmatrix} \tag{10}$$

can be gauged transformed to the following form:

$$L(z,S)=\frac{1}{z}\times$$
$$\times\begin{pmatrix} S_{11}-z^2S_{12} & S_{12}\\ S_{21}-z^2(S_{11}-S_{22})-z^4S_{12} & S_{22}+z^2S_{12}\end{pmatrix}. \tag{11}$$

The residue matrix is given by the following change of variables:

$$S=\begin{pmatrix} S_{11} & S_{12}\\ S_{21} & S_{22}\end{pmatrix}=\begin{pmatrix} \frac{1}{2}v\,u & -\frac{1}{2}\frac{v}{u}\\ \frac{1}{2}(v\,u^3-2\nu u^2) & -\frac{1}{2}v\,u+\nu\end{pmatrix}, \tag{12}$$

i.e. the canonical variables v,u are transformed into the generators of the Lie algebra gl_2 with the Poisson–Lie brackets:

$$\{S_{ij},S_{kl}\}=\delta_{il}S_{kj}-\delta_{kj}S_{il}. \tag{13}$$

The Hamiltonian (9) acquires the form

$$\begin{aligned} H&=-S_{12}(S_{11}-S_{22})=\frac{1}{2}\,\mathrm{tr}\,[S\,J(S)]\,,\\ J(S)&=-\begin{pmatrix} S_{12} & 0\\ S_{11}-S_{22} & -S_{12}\end{pmatrix}\end{aligned} \tag{14}$$

of the integrable (rational) top of Euler–Arnold type with the inverse inertia tensor $J(S)$. Equations of motion

$$\dot S=\{H,S\}=[S,J(S)], \tag{15}$$

can be written in the Lax form (1) with the Lax matrix (11) and the M-matrix

$$\mathcal{M}(z,S)=-\begin{pmatrix} S_{12} & 0\\ S_{11}-S_{22}+2z^2S_{12} & -S_{12}\end{pmatrix}. \tag{16}$$

The top form of the Calogero–Moser model allows us to relate to it the non-dynamical r-matrix. Indeed, the classical r-matrix provides the Poisson brackets between matrix elements of the Lax matrix in the form:

$$\begin{aligned}&\sum_{i,j,k,l}E_{ij}\otimes E_{kl}\,\{L_{ij}(z),L_{kl}(w)\}:=\\ &=\{L_1(z),L_2(w)\}=[L_1(z)+L_2(w),r_{12}(z-w)],\end{aligned} \tag{17}$$

where for gl_2: $L_1 = L\otimes 1 = \begin{pmatrix} L_{11}\,1_{2\times2} & L_{12}\,1_{2\times2} \\ L_{21}\,1_{2\times2} & L_{22}\,1_{2\times2}\end{pmatrix}$, $L_2 = 1\otimes L = \begin{pmatrix} L & 0_{2\times2} \\ 0_{2\times2} & L\end{pmatrix}$. In our example the classical r-matrix equals

$$r_{12}(z) = \begin{pmatrix} 1/z & 0 & 0 & 0 \\ -z & 0 & 1/z & 0 \\ -z & 1/z & 0 & 0 \\ -z^3 & z & z & 1/z \end{pmatrix}. \qquad (18)$$

It satisfies the classical Yang–Baxter equation

$$[r_{12}(z-w), r_{13}(z)] + [r_{12}(z-w), r_{23}(w)] + \\ + [r_{13}(z), r_{23}(w)] = 0 \qquad (19)$$

and is simply related to the Lax matrix:

$$L(z) = \mathrm{tr}_2\,[r_{12}(z)S_2]\,. \qquad (20)$$

3. Relativistic models and quantum *R*-matrices. As it was shown in [3] the construction of the top model can be generalized to the relativistic deformation of integrable systems. The simplest example here is the rational 2-body Ruijsenaars–Schneider model. It is described by the Hamiltonian

$$H^{\mathrm{RS}} = \frac{2u-\eta}{2u}e^{v/c} + \frac{2u+\eta}{2u}e^{-v/c}, \qquad (21)$$

where η is the coupling constant and c is the light speed. As in the previous case the Ruijsenaars–Schneider model can be rewritten in the form of the (relativistic) top. The Lax matrix has the following form

$$L^\eta(z,\mathcal{S}) = \frac{1}{z}\mathcal{S}_{2\times2} + \frac{\mathrm{tr}(S)}{\eta}1_{2\times2} - (z+\eta)\times \\ \times \begin{pmatrix} S_{12} & 0 \\ (\mathcal{S}_{11}-\mathcal{S}_{22}) + (\eta^2+z^2+\eta z)\mathcal{S}_{12} & -\mathcal{S}_{12}\end{pmatrix} \qquad (22)$$

with the change of variables

$$\mathcal{S}_{11}(v,u) = -\frac{u}{2}\left(e^{v/c} - e^{-v/c}\right), \\ \mathcal{S}_{12}(v,u) = \frac{1}{2u}\left(e^{v/c} - e^{-v/c}\right), \\ \mathcal{S}_{21}(v,u) = -\frac{u}{2}\left[e^{v/c}(u-\eta)^2 - e^{-v/c}(u+\eta)^2\right], \\ \mathcal{S}_{22}(v,u) = \frac{1}{2u}\left[e^{v/c}(u-\eta)^2 - e^{-v/c}(u+\eta)^2\right] \qquad (23)$$

and equation of motion

$$\dot{\mathcal{S}} = [\mathcal{S}, J^\eta(\mathcal{S})], \quad J^\eta(S) = \frac{\mathrm{tr}\mathcal{S}}{\eta}\,1_{2\times2} - \\ - \begin{pmatrix} \eta\mathcal{S}_{12} & 0 \\ \eta^3\mathcal{S}_{12} + \eta(\mathcal{S}_{11}-\mathcal{S}_{22}) & -\eta\mathcal{S}_{12}\end{pmatrix} \qquad (24)$$

generated by the Lax equations (1) with (22) and $M(z,\mathcal{S}) = -L(z,\mathcal{S})$ (i.e. the M-matrix here is of the same form as the L-matrix (11) in the non-relativistic case up to the sign).

The Poisson brackets are defined by the quadratic r-matrix structure

$$\{L_1^\eta(z)\,, L_2^\eta(w)\} = [L_1^\eta(z)\,L_2^\eta(w), r_{12}(z-w)], \qquad (25)$$

with the rational r-matrix (18). The quadratic Poisson brackets for the matrix elements of $\mathcal{S}$ is the classical Sklyanin algebra:

$$\{\mathcal{S}_1, \mathcal{S}_2\} = [J^\eta(S)_1\,\mathcal{S}_2, P_{12}]. \qquad (26)$$

The most important statement here is the following: while the non-relativistic top is described by the classical r-matrix (20) the relativistic top is related in the same way to the quantum R-matrix:

$$L^\eta(z,\mathcal{S}) = \sum_{i,j,k,l=1}^{N} R^\eta_{ij,kl}(z)\,E_{ij}\,\mathcal{S}_{lk} = \mathrm{tr}_2\,[R^\eta_{12}(z)\mathcal{S}_2]\,, \qquad (27)$$

where the relativistic deformation parameter η plays the role of the Planck constant. That is to say that $R^\eta_{12}(z)$ satisfies the quantum Yang–Baxter equation:

$$R^\eta_{12}(z-w)R^\eta_{13}(z-y)R^\eta_{23}(w-y) = \\ = R^\eta_{23}(w-y)R^\eta_{12}(z-w)R^\eta_{13}(z-y). \qquad (28)$$

With a knowledge of the Lax matrix we know the quantum R-matrix as well. For the case (22) we have:

$$R^\hbar(z) = \\ \begin{pmatrix} \hbar^{-1}+z^{-1} & 0 & 0 & 0 \\ -\hbar-z & \hbar^{-1} & z^{-1} & 0 \\ -\hbar-z & z^{-1} & \hbar^{-1} & 0 \\ -\hbar^3-2z\hbar^2-2\hbar z^2-z^3 & \hbar+z & \hbar+z & \hbar^{-1}+z^{-1}\end{pmatrix}. \qquad (29)$$

The classical limit

$$R^\eta_{12}(z) = \hbar^{-1}1\otimes 1 + r_{12}(z) + \hbar\,m_{12} + O(\hbar^2) \qquad (30)$$

provides the classical Yang-Baxter equation (19) from the quantum one (28) and corresponds to the non-relativistic limit at the level of mechanical systems:

$$\eta := \nu/c, \quad c\to\infty:\ H^{\mathrm{RS}} = \\ = -\eta^{-1}\,\mathrm{tr}\mathcal{S}(v,u) = 2 + \frac{2}{c^2}H^{\mathrm{CM}} + o\left(\frac{1}{c^2}\right) \qquad (31)$$

and $S(v,u) = -\frac{1}{2}\lim\limits_{c\to\infty} c\,\mathcal{S}(v,u)$.

4. Spin chains and Gaudin models. After integrable many-body systems are included into the class of integrable tops we can proceed to more complicated models. Having the quadratic Poisson structure (25) the classical periodic spin chain with n sites is naturally defined via the monodromy matrix

$$T(z,\mathcal{S}^1,...,\mathcal{S}^n)=T(z)=L^\eta(z-z_1,\mathcal{S}^1)\,...\,L^\eta(z-z_n,\mathcal{S}^n), \tag{32}$$

where z_k are the inhomogeneities parameters. In view of (27) it takes the form:

$$T_0(z)=\mathrm{tr}_{1...n}\left[R^\eta_{01}(z-z_1)\,...\,R^\eta_{0n}(z-z_n)\,(\mathcal{S}^1)_1\,...\,(\mathcal{S}^n)_n\right], \tag{33}$$

where the index "0" corresponds to the common matrix space (auxiliary space) of Lax matrices $L^\eta(\mathcal{S}^n)$. We drop it in formulae below. In the non-relativistic limit $\eta\to 0$ the monodromy matrix (33) gives rise to the Lax operator of the Gaudin model:

$$L^{\mathrm{G}}(z)=\sum_{a=1}^n \mathrm{tr}_a\left[r_{0a}(z-z_a)S^a\right]\overset{(20)}{=}\sum_{a=1}^n L(z-z_a,S^a). \tag{34}$$

The Poisson structures for both – spin chains (32) and Gaudin models (34), (33) are direct sums oven $a=1,...,n$ of 1-site Poisson structures (26), and (13) respectively. For example, the Poisson structure for the Gaudin model is

$$\{S_1^a,S_2^b\}=\delta^{ab}\,[S_2^a,P_{12}]. \tag{35}$$

The Hamiltonians appear by evaluating

$$\frac{1}{2}\mathrm{tr}\left[L^{\mathrm{G}}(z)\right]^2=\frac{1}{2}\sum_{a=1}^n\frac{\mathrm{tr}\,(S^a)^2}{(z-z_a)^2}-\frac{h_a}{z-z_a}+2h_0. \tag{36}$$

The direct computation gives

$$\begin{aligned} h_a&=-\sum_{c\neq a}^n \mathrm{tr}_{12}\left[r_{12}(z_a-z_c)S_1^aS_2^c\right],\\ h_0&=\frac{1}{2}\sum_{b,c=1}^n \mathrm{tr}\left[S^b\,\mathcal{M}(z_b-z_c,S^c)\right], \end{aligned} \tag{37}$$

where $\mathcal{M}(z,S)$ is the M-operator (16). The Hamiltonians (37) generate equations of motion

$$\begin{cases} \partial_{t_a}S^b=-[S^b,L(z_a-z_b,S^a)], & b\neq a=1,...,n,\\ \partial_{t_a}S^a=\sum\limits_{c\neq a}^n[S^a,L(z_c-z_a,S^c)], & a=1,...,n \end{cases} \tag{38}$$

and

$$\partial_{t_0}S^a=[S^a,J(S^a)]+\sum_{c\neq a}[S^a,\mathcal{M}(z_a-z_c,S^c)]. \tag{39}$$

Eqs. (38) and (39) have the Lax form

$$\partial_{t_d}L^{\mathrm{G}}(z)=[L^{\mathrm{G}}(z),M_d^{\mathrm{G}}],\quad d=0,1,...,n \tag{40}$$

with $M_a^{\mathrm{G}}(z)=-L(z-z_a,S^a)$, $a=1,...,n$, and $M_0^{\mathrm{G}}(z)=\sum\limits_{c=0}^n\mathcal{M}(z-z_c,S^c)$, $\mathcal{M}(z,S)$ is from (16).

5. Soliton equations. For the homogeneous spin chain (33) ($z_k=0$) the continuous limit leads to the 1+1 field theories, which are integrable in the sense of the classical inverse scattering method. The equations of motion are defined by the Zakharov–Shabat equations (4):

$$\partial_t U-k\partial_x V=[U,V], \tag{41}$$

where U and V are gl_2-valued functions on the circle (with the coordinate x). They also depend on the spectral parameter and dynamical fields $S(x)$. The mechanical (0+1) models described by non-dynamical r-matrix can be generalized to 1+1 field theory (41) straightforwardly: one should simply use the same Lax operator as for the top model:

$$U^{\mathrm{LL}}[z,S(x)]=L[z,S(x)]=\mathrm{tr}_2[r_{12}(z)S_2(x)]. \tag{42}$$

It leads to Landau–Lifshitz type equation (the components of $S(x)$ in the Pauli matrices basis $S(x)==\sum_1^3\sigma_aS_a(x)$ can be considered as components of the magnetization vector in one dimensional ferromagnetic):

$$\partial_t S=\alpha[S,S_{xx}]+[S,J(S)], \tag{43}$$

where $S_{xx}=\partial_x^2S$, $J(S)$ is the same as in the top case and $\alpha=\lambda^2/(8k^2)$ is a constant ($S^2=\lambda^2\,1$, $\partial_x\lambda=0$). The equations of motion (43) with $J(S)$ (14) are of the form:

$$\begin{cases} \partial_tS_{11}=\alpha S_{12}\partial_x^2S_{21}-\alpha S_{21}\partial_x^2S_{12}-2S_{12}S_{11},\\ \partial_tS_{21}=2\alpha S_{21}\partial_x^2S_{11}-2\alpha S_{11}\partial_x^2S_{21}-2S_{12}S_{21}+4S_{11}^2,\\ \partial_tS_{12}=2\alpha S_{11}\partial_x^2S_{12}-2\alpha S_{12}\partial_x^2S_{11}+2S_{12}^2 \end{cases} \tag{44}$$

are described by the Hamiltonian

$$H^{\mathrm{LL}}=\frac{1}{2}\oint dx\left\{\mathrm{tr}(S_x^2)+\mathrm{tr}[SJ(S)]\right\} \tag{45}$$

and are presented in the form (41). The matrix V is obtained as follows:

$$V^{\mathrm{LL}}=-[z^{-1}L(z,S)-2\mathcal{M}(z,S)+L(z,h)]/2, \tag{46}$$

where L and $\mathcal{M}$ are from (11), (16) and matrix h is equal to $h=-\frac{k}{4\lambda^2}[S,S_x]$, $S_x=\partial_xS$.

Similarly to the Gaudin model, we can construct the 1+1 theory corresponding to $U(z)$ with two poles:

$$U^{\rm chiral}[z,S^1(x),S^2(x)]=L[z-z_1,S^1(x)]+L[z-z_2,S^2(x)]. \quad (47)$$

It is the principal chiral model. Equations of motion are of the form:

$$\begin{cases}\partial_t S^1-k\partial_x S^1=-2[S^1,L(z_1-z_2,S^2)],\\ \partial_t S^2+k\partial_x S^2=-2[S^2,L(z_1-z_2,S^1)].\end{cases} \quad (48)$$

6. Painlevé–Calogero correspondence. The six Painlevé equations were discovered in the 1900–1910 period as the second order differential equations that have only poles in the complex plane as movable singularities. The most general equation – the Painlevé VI (PVI) has the following form:

$$\frac{d^2X}{dt^2}=\frac{1}{2}\left(\frac{1}{X}+\frac{1}{X-1}+\frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2-$$
$$-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{X-t}\right)\frac{dX}{dt}+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\times$$
$$\times\left(\alpha+\beta\frac{t}{X^2}+\gamma\frac{t-1}{(X-1)^2}+\delta\frac{t(t-1)}{(X-t)^2}\right), \quad (49)$$

where $(\alpha,\beta,\gamma,\delta)$ are arbitrary complex constants. The PVI can be represented in the elliptic form

$$\frac{d^2u}{d\tau^2}=\sum_{a=0}^{3}\nu_a^2\wp'(u+\omega_a),\ (\omega_0,\omega_1,\omega_2,\omega_3)=$$
$$=\left(0,\frac{1}{2},\frac{\tau+1}{2},\frac{\tau}{2}\right) \quad (50)$$

via the change of variables $X=\frac{\wp(u)-e_1}{e_2-e_1}$, $t=\frac{e_3-e_1}{e_2-e_1}$ with $e_k=\wp(\omega_k)$ and identification of constants $(\nu_0^2,\nu_1^2,\nu_2^2,\nu_3^2)=\frac{1}{(2\pi i)^2}(\alpha,-\beta,\gamma,\frac{1}{2}-\delta)$. Here $\wp(x)$ is the Weierstrass $\wp$-function on the elliptic curve $\Sigma_\tau=\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$, $\mathrm{Im}\,\tau>0$. Eq. (50) is described as the Hamiltonian system with the Hamiltonian function $H=\frac{1}{2}v^2-\sum_{a=0}^{3}\nu_a^2\wp(u+\omega_a)$ and canonical Poisson bracket $\{v,u\}=1$. The system is *non-autonomous* since the potential explicitly depends on the moduli τ (of Σ_τ) which is the "time" variable. It is non-autonomous version of the BC_1 Calogero-Inozemtsev system. In the case when $\nu_a=\frac{1}{2}\nu$ for all a we come to the elliptic two-particle non-autonomous Calogero–Moser model

$$\frac{d^2u}{d\tau^2}=\nu^2\wp'(2u). \quad (51)$$

Similarly to the rational case the model (50) has 2×2 Lax pair [4] and can be formulated in the form of integrable top [5]. It is the non-autonomous version of the Zhukovsky–Volterra gyrostat. Namely, it was shown in [5] that (50) can be written as dynamics of three-dimensional complex-valued vector $\mathbf{S}=(S_1,S_2,S_3)$:

$$\partial_\tau\mathbf{S}=\mathbf{S}\times\mathbf{J}(\mathbf{S})+\mathbf{S}\times\boldsymbol{\nu}', \quad (52)$$

where $\mathbf{J}(\mathbf{S})=(J_1S_1,J_2S_2,J_3S_3)$, $J_k=\wp(\omega_k)$, and $\boldsymbol{\nu}'=(\nu_1',\nu_2',\nu_3')$ – vector of linear combinations of constants ν_a from (50) multiplied by some ratios of theta-constants. The fourth independent linear combination of the constants $\nu_0'=\frac{1}{2}\sum_{c=0}^{3}\nu_c$ appears to be the length of $\mathbf{S}$: $\nu_0'^2=\sum_{\alpha=1}^{3}S_\alpha^2$. Eq. (52) can be also written in terms of $\mathrm{Sl}(2,\mathbb{C})$-valued matrix $S=\sum_{\gamma=1}^{3}\sigma_\gamma S_\gamma$, where σ_γ are the Pauli matrices, as

$$\partial_\tau S=[S,J(S)]+[S,\nu'], \quad (53)$$

where $\nu'=\sum_{\gamma=1}^{3}\sigma_\gamma\nu_\gamma'$. It is generated by the quadratic Hamiltonian

$$H=\frac{1}{2}\mathrm{tr}\,[S\,J(S)]+\mathrm{tr}(S\,\nu'), \quad (54)$$

and the linear Poisson–Lie brackets on $\mathrm{sl}^*(2,\mathbb{C})$: $\{S_\alpha,S_\beta\}=\varepsilon_{\alpha\beta\gamma}S_\gamma$. Explicit change of variables $S_\alpha(v,u)$ and other details can be found in [5–7]. Eq. (53) is reduced to the non-autonomous elliptic sl_2-top when $\nu_{1,2,3}'=0$. This case corresponds to (51).

The (classical) Painlevé–Calogero correspondence was suggested in [8]. It claims that the (Krichever's) Lax pair of the elliptic Calogero–Moser model can be also used for the monodromy preserving equations, which describe the higher rank Painlevé equations in the elliptic form.

The Painlevé–Calogero correspondence can be formulated as the following property of the quantum non-dynamical R-matrix:

$$\partial_\tau R^\hbar(z)=\partial_z\partial_\hbar R^\hbar(z). \quad (55)$$

Here we imply explicit dependence $R=R(\tau)$. For the elliptic case the quantum R-matrix is the Baxter–Belavin's one. Plugging the expansion (30) of the classical limit into (55) we get a set of relations. The first non-trivial is

$$\partial_\tau r_{12}^\tau(z)=\partial_z m_{12}^\tau(z), \quad (56)$$

where $r_{12}^\tau(z)=r_{12}(z,\tau)$ is the classical r-matrix. From (56) it follows that

$$\partial_\tau L^\tau(z,S)=\partial_z\mathcal{M}^\tau(z,S), \quad (57)$$

where $L^\tau(z,S) = L(z,S,\tau)$, $\mathcal{M}^\tau(z,S) = \mathcal{M}(z,S,\tau)$, and the derivative is taken with respect to explicit dependence on τ. Therefore, we can consider the monodromy preserving equations in time τ

$$\begin{aligned} d_\tau L^\tau(z,S) - \partial_z \mathcal{M}^\tau(z,S) = \\ = [L^\tau(z,S), \mathcal{M}^\tau(z,S)], \quad S = S(\tau) \end{aligned} \tag{58}$$

($d_\tau = d/d\tau$) as the non-autonomous version of the integrable top's equations of motion:

$$\partial_\tau S = [S, J^\tau(S)]. \tag{59}$$

Indeed, the total derivative $d_\tau L^\tau(z,S)$ contains both – the partial derivatives by explicit and implicit dependence on τ:

$$\begin{aligned} d_\tau L^\tau[z,S(\tau)] = d_\tau \mathrm{tr}_2[r^\tau_{12}(z) S_2] = \\ \mathrm{tr}_2\Big\{[\partial_\tau r^\tau_{12}(z)]\, S_2\Big\} + \mathrm{tr}_2\Big[r^\tau_{12}(z)\,(\partial_\tau S_2)\Big]. \end{aligned} \tag{60}$$

The first term is cancelled by $\partial_z \mathcal{M}^\tau(z,S)$ (57), and we get (59).

Similarly, the Schlesinger system as the non-autonomous Gaudin model [9]. The monodromy preserving equations

$$\partial_{z_a} L^{\mathrm{G}}(z) - \partial_z M^{\mathrm{G},\,a}(z) = [L^{\mathrm{G}}(z), M^{\mathrm{G},\,a}(z)] \tag{61}$$

and

$$\partial_\tau L^{\mathrm{G}}(z) - \partial_z M^{\mathrm{G},\,0}(z) = [L^{\mathrm{G}}(z), M^{\mathrm{G},\,0}(z)] \tag{62}$$

generate dynamics in time variables z_a and τ. They are equivalent to non-autonomous versions of the Gaudin's equations of motion (38), (39):

$$\begin{cases} \partial_{z_a} S^b = -[S^b, L^\tau(z_a - z_b, S^a)], \ b \neq a, \\ \partial_{z_a} S^a = \displaystyle\sum_{c\neq a}^{n} [S^a, L^\tau(z_c - z_a, S^c)] \end{cases} \tag{63}$$

for $a, b = 1, ..., n$ and

$$\partial_\tau S^a = [S^a, J^\tau(S^a)] + \sum_{c\neq a} [S^a, \mathcal{M}^\tau(z_a - z_c, S^c)]. \tag{64}$$

The Hamiltonians and the Poisson structure of the Schlesinger system are the same as for the Gaudin model. Description of the elliptic Schlesinger system can be found in [9].

7. Quantum Painlevé-Calogero correspondence. The classical Painlevé-Calogero correspondence (55)–(57) can be generalized to the quantum one in the following sense (see [10]). The monodromy preserving equation (2) is the compatibility condition of the linear (isomonodromy) problem:

$$\begin{cases} \partial_z \mathbf{\Psi} = -L(z,t)\mathbf{\Psi}, \\ \partial_t \mathbf{\Psi} = -M(z,t)\mathbf{\Psi}, \end{cases} \quad \mathbf{\Psi} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{65}$$

The main statement is that for all (2×2 linear problems) Painlevé equations there exists a choice of coordinates z, t and gauge fixation (6) such the linear problem (65) is reduced to the scalar equation on ψ_1

$$\partial_t \psi_1 = \left[\frac{1}{2}\,\partial_z^2 + \frac{1}{2}\,\det L(z) - \frac{1}{2}\,\partial_z L_{11}(z) + M_{11}(z)\right]\psi_1, \tag{66}$$

and (66) describes the non-stationary Schrödinger equation for the potential of the classical mechanical model – the classical Painlevé equation coming from (2). The quantum potential may differ from the classical by only "quantum shift" of the constants.

Let us give the example of the truncated $\mathrm{P_{III}}$ equation:

$$\ddot{u} = 2\nu^2 e^t \sinh(2u). \tag{67}$$

The linear problem was found in [11]:

$$L(z,t) = \begin{pmatrix} \dot{u} & 2\nu e^{t/2}\sinh(z-u) \\ 2\nu e^{t/2}\sinh(z+u) & -\dot{u} \end{pmatrix}, \tag{68}$$

$$M(z,t) = \begin{pmatrix} 0 & \nu e^{t/2}\cosh(z-u) \\ \nu e^{t/2}\cosh(z+u) & 0 \end{pmatrix}. \tag{69}$$

Then, we have the following quantization of (67):

$$\begin{aligned} \partial_t \psi_1 = \Big[H_{\mathrm{III}}(\partial_z, z) - H_{\mathrm{III}}(\dot{u}, u)\Big]\psi_1, \\ H_{\mathrm{III}}(\partial_z, z) = \frac{1}{2}\,\partial_z^2 - \nu^2 e^t \cosh(2z). \end{aligned} \tag{70}$$

8. Knizhnik–Zamolodchikov–Bernard equations. The KZB equation can be defined as quantization of the linear problem (65) for the Schlesinger system (61)–(64):

$$\begin{cases} \hat{\nabla}_a \psi = 0, \\ \hat{\nabla}_\tau \psi = 0, \end{cases} \tag{71}$$

where

$$\begin{aligned} \nabla_a = \partial_{z_a} + \sum_{c\neq a} r_{ac}(z_a - z_c), \\ \nabla_\tau = \partial_\tau + \frac{1}{2}\sum_{b,c} m_{bc}(z_b - z_c) \end{aligned} \tag{72}$$

with r and m are from (56). The compatibility condition of (71) requires classical Yang–Baxter equation (19), heat equation (55), and

$$[r_{ab}, m_{ab}] = 0, \tag{73}$$

$$[r_{ab}, m_{ac} + m_{bc}] + [r_{ac}, m_{ab} + m_{bc}] = 0. \tag{74}$$

It appears [12] that these identities can be proved in the general form using the quantum Yang–Baxter equation (28) and another one:

$$R^{\hbar}_{12} R^{\hbar}_{23} R^{\hbar}_{31} + R^{\hbar}_{13} R^{\hbar}_{32} R^{\hbar}_{21} = -N^3 \wp'(\hbar)\, 1\otimes 1\otimes 1, \tag{75}$$

where $R^{\hbar}_{ab} = R^{\hbar}_{ab}(z_a - z_b)$.

Therefore, we get the quantization of the Schlesinger system as consequence of the underlying quantum R-matrix properties. The quantum R-matrix structure underlies classical integrable systems including their relativistic extensions and Painlevé-Schlesinger equations as well.

9. Painlevé field theory. To construct field-theoretical generalization of the Painlevé equation (52) or (53) we replace the linear system (3) defined on Σ by the four-dimensional linear system over $\Sigma \times \tilde{\Sigma}$, where $\tilde{\Sigma}$ is $\mathbb{C}^*$ parameterized by x, or a non-commutative torus. The first case corresponds to the equations of type (5). In particular, for field generalization of the Painlevé equation the constants $\boldsymbol{\nu}'$ in (53) become independent fields $S^{\alpha}_{b}(x)$ and the inverse inertia tensor J now is some pseudo-differential operator. Though the field theory is non-local, the non-locality is similar to the non-locality of the hydrodynamic of the ideal fluid in terms of vorticity. In fact, it is non-autonomous version of the hydrodynamic, where the Laplace operator is replaced by some pseudo-differential operator. If the initial value problem for this system depends on the zero modes only, then the equation coincides with the Painlevé equation (52).

In the second case in general situation we also come to non-local equations, but some degenerate cases lead to local equations. For example, in the quasi-classical limit of the non-commutative torus and the rational degeneration of Σ we obtain the third order equation in the 2+1 space (τ, x_1, x_2), depending on two parameters ϵ_1, ϵ_2

$$\partial_\tau \bar{\partial}_Z^2 \mathbf{F}(x,\tau) - \{\bar{\partial}_Z^2 \mathbf{F}(x,\tau), \mathbf{F}(x,\tau)\} + \epsilon_2 \partial_{x_2} \bar{\partial}_Z \mathbf{F}(x,\tau) = 0,$$

where

$$\bar{\partial}_Z = \frac{1}{2\pi i}(\epsilon_1 \partial_{x_1} + \epsilon_2 \tau \partial_{x_2}),$$

$$\{f, g\} = \partial_{x_1} f \partial_{x_2} g - \partial_{x_2} f \partial_{x_1} g.$$

Again, for the zero modes in this equation is a special degeneration of the Painlevé equation.

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