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Clifford numbers from Bohr–Sommerfeld quantization of Grassmann-variant systems

S. Abe[+*1)], D. A. Tayurskii[*×]

[+]Department of Physical Engineering, Mie University, 514-8507 Mie, Japan

[*]Institute of Physics, Kazan Federal University, 420008 Kazan, Russia

[×]Centre for Quantum Technologies, Kazan Federal University, 420008 Kazan, Russia

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Bohr–Sommerfeld quantization is discussed for Grassmann-variant fermionic degrees of freedom. It is shown that this procedure applied to a Grassmannian dynamical variable of the Majorana type gives rise to a Clifford number, shedding further light on the quantum-classical correspondence for fermions.

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Fermions are always of interest in view of the correspondence principle, because of the subtleties of their counterparts in classical theory. An approach to this point is to introduce into systems Grassmannian variables that are nilpotent. Today, it is standard to employ such variables in path-integral quantization of fermions. However, to intuitively understand such degrees of freedom, it is of importance to clarify what the quantum-classical correspondence between fermions and such exotic variables is. In this context, Berezin's celebrated proposal [1] of calculus of Grassmannian variables has been a cornerstone, and the subsequent work of Berezin and Marinov [2] (see also [3]) on quantization of Grassmann-variant classical systems has played a prominent role for understanding how the correspondence principle can be established for fermions. However, it may be fair to say even today that these investigations still remain formal since the exact quantization procedure itself is inevitably so [4].

Obviously, Bohr–Sommerfeld quantization in early quantum mechanics can shed light on the correspondence principle in a peculiar manner, since it shows in the minimal way how the classical concepts should be modified. This point has ever been addressed for fermions in the literature [5, 6]. There, Bohr–Sommerfeld quantization of fermionic degrees of freedom has been proposed with the use of the Berezin calculus.

Consider a time-dependent Grassmann-variant vector $\boldsymbol{\xi}(t) = (\xi_1(t), \xi_2(t), \xi_3(t))$, each component of which is nilpotent: $\xi_i^2(t) = 0$, implying the anticommutativity $\xi_i(t)\xi_j(t) = -\xi_j(t)\xi_i(t)$ $(i, j = 1, 2, 3)$. It is real Grassmannian in the sense that $\xi_i^*(t) = \xi_i(t)$. Let us use this vector to construct the following Lagrangian:

$$L = \frac{i}{2}\boldsymbol{\xi} \cdot \frac{d\boldsymbol{\xi}}{dt} + \frac{i}{2}\mathbf{B} \cdot (\boldsymbol{\xi} \times \boldsymbol{\xi}), \tag{1}$$

where $\mathbf{B}$ is an ordinary real constant vector. This system is analogous to the harmonic oscillator in an ordinary bosonic theory and turns out to be useful for describing our semiclassical scheme. The canonical momentum Π_i conjugate to ξ_i is given by $\Pi_i = [\partial/\partial\,(d\xi_i/dt)]\,L = -(i/2)\xi_i$, which yields the second-class constraint in Dirac's terminology [7]. (The rule of the differentiation with respect to Grassmann variables η and ζ is: $(\partial/\partial\zeta)\,(\eta\zeta) = (\partial/\partial\zeta)\,(-\zeta\eta) = -\eta$, for example.) The Hamiltonian obtained by the Legendre transformation of the Lagrangian in Eq. (1), $H = (d\xi_i/dt)\,\Pi_i - L$ (the summation convention to be understood for the repeated indices), reads $H = -(i/2)\mathbf{B} \cdot (\boldsymbol{\xi} \times \boldsymbol{\xi})$. Within the framework of Dirac's generalized canonical formalism [7] and the canonical quantization condition (in the unit $\hbar = 1$), the quantum-mechanical operators $\hat{\xi}_i$'s are found to satisfy the following anticommutation relation [2, 3]:

$$\hat{\xi}_i\hat{\xi}_j + \hat{\xi}_j\hat{\xi}_i = \delta_{ij}, \tag{2}$$

which results the identification: $\hat{\xi}_i = \sigma_i/\sqrt{2}$, where σ_i's are the Pauli matrices. Thus, the Hamiltonian is quantized as $\hat{H} = (1/2)\boldsymbol{\sigma} \cdot \mathbf{B}$, which describes the Pauli spin in the magnetic field $\mathbf{B}$. Its energy eigenvalues are the familiar ones: $E_\pm = \pm B/2$ with $B = |\mathbf{B}|$.

Let us consider the above system from the viewpoint of Bohr–Sommerfeld quantization. We define

[1)]e-mail: suabe@sf6.so-net.ne.jp

the tensor dual to B_k ($k = 1,2,3$), i.e., $\tilde{B}_{ij} = \varepsilon_{ijk}B_k$. This antisymmetric 3×3 matrix is diagonalized by a time-independent unitary matrix, S: $S^\dagger \tilde{B} S = \mathrm{diag}(-iB, iB, 0)$. The corresponding normal coordinates, $\psi_i = (S^\dagger)_{ij}\xi_j$, satisfy the relations: $\psi_2 = \psi_1^*$ and $\psi_3 = \psi_3^*$. They are explicitly given as follows:

$$\psi_1 = \frac{B_1B_3 - iB_2B}{B\sqrt{2(B_1^2+B_2^2)}}\xi_1 + \frac{B_2B_3 + iB_1B}{B\sqrt{2(B_1^2+B_2^2)}}\xi_2 - \frac{\sqrt{B_1^2+B_2^2}}{\sqrt{2}B}\xi_3, \tag{3}$$

$$\psi_3 = \frac{B_i\xi_i}{B}. \tag{4}$$

Thus, $\psi \equiv \psi_2$ and $\chi \equiv \psi_3/\sqrt{2}$ are complex and real Grassmannian variables, respectively. The rule of complex conjugation is that $(\psi_i\psi_j)^* = \psi_j^*\psi_i^* = -\psi_i^*\psi_j^*$. The Lagrangian in Eq. (1) is then rewritten as follows:

$$L = \psi^*\left(i\frac{d}{dt} - B\right)\psi + i\chi\frac{d\chi}{dt}. \tag{5}$$

It is noted that ψ is a (0+1)-dimensional analogue of the Dirac field with “mass” B, whereas χ is the Majorana-like field with vanishing mass: χ cannot have a mass term, since classically $\chi^2 = 0$. The canonical momenta conjugate to ψ, ψ^*, and χ are given by

$$\Pi_\psi = -i\psi^*, \tag{6}$$

$$\Pi_{\psi^*} = 0, \tag{7}$$

$$\Pi_\chi = -i\chi, \tag{8}$$

respectively. The classical Hamiltonian, $H = (d\psi/dt)\Pi_\psi + (d\psi^*/dt)\Pi_{\psi^*} + (d\chi/dt)\Pi_\chi - L$, reads

$$H = B\psi^*\psi, \tag{9}$$

which does not contain χ. The equations of motion for ψ and χ are

$$\left(i\frac{d}{dt} - B\right)\psi = 0, \tag{10}$$

$$\frac{d\chi}{dt} = 0, \tag{11}$$

which can also be obtained within the framework of Dirac's generalized canonical formalism for Grassmannian dynamical variables. Eq. (10) has the periodic solution

$$\psi(t) = \psi(0)e^{-iBt}, \tag{12}$$

with the period $2\pi/B$. Bohr–Sommerfeld quantization of the complex Grassmannian variable ψ [5, 6] is based on the two equivalent forms of the phase-space quantity $d\psi\Pi_\psi$, that is,

$$-id\psi\psi^* = -idt\frac{d\psi}{dt}\psi^*. \tag{13}$$

Integrating the both sides of this equation over a single period, we have

$$-i\int_0^{2\pi/B} dt\frac{d\psi}{dt}\psi^* = -i\int d\psi\psi^*. \tag{14}$$

These quantities can be thought of as the Ehrenfest adiabatic invariant. It is noted that at the stage of the integrations in Eq. (14), the dynamical variables must already be regarded as quantum-mechanical ones, i.e., q-numbers in Dirac's terminology [4]. That is, the original Grassmannian nilpotent nature should be replaced by the quantum analogue. The left-hand side of Eq. (14) is evaluated for the periodic classical solution in Eq. (12), yielding $2\pi\psi^*(0)\psi(0)$. On the other hand, the right-hand side should be regarded as the Berezin integration [1, 6]:

$$-i\int d\psi\ \psi^* = \pm\pi, \tag{15}$$

that is, the integrations over the original real Grassmannian variables ξ's are pure imaginary [8]

$$\int d\xi_i\xi_j = \pm i\pi\delta_{ij}, \tag{16}$$

where the double signs reflect the fact that $d\xi_i\xi_i$ is *only known to be pure imaginary* under the rule: $(d\xi_i\xi_i)^* = \xi_i d\xi_i = -d\xi_i\xi_i$. The value of the integral in Eqs. (16) has to be kept unchanged throughout the discussion. Eqs. (15) and (16) give an important physical insight: in contrast to a “bosonic” system in ordinary phase space (such as the hydrogen atom or the harmonic oscillator, for example), excitation of a fermionic system is finite, and the Berezin integrations in Eqs. (15) and (16) mathematically assure this feature. Consequently, it follows that

$$\psi^*\psi = \pm\frac{1}{2}, \tag{17}$$

giving rise to the exact quantization of the energy in Eq. (9):

$$H = E_\pm = \pm\frac{1}{2}B. \tag{18}$$

Now, the remaining subject, which is the main one of the present work, is to formulate Bohr–Sommerfeld quantization of the Majorana type satisfying Eq. (11). This will turn out to contain a subtle point regarding the “massless” nature of χ. The phase-space quantity $d\chi\Pi_\chi$ has again the following two equivalent forms:

$$-id\chi\chi = -idt\frac{d\chi}{dt}\chi. \tag{19}$$

Naively, the right-hand side of this equation might be zero because of the classical equation of motion in Eq. (11), whereas the integral of the left-hand side is non-zero because of the rule of the Berezin integration, leading to an apparent contradiction. However, it is crucial to note that motion at rest can be regarded as a periodic motion with an infinite period. Therefore, the periodic classical solution of Eq. (11) and its derivative could be understood as

$$\chi(t) = \chi(0)e^{i2\pi t/T} \mid_{T\to\infty}, \tag{20}$$

$$\frac{d\chi(t)}{dt} = \frac{i2\pi}{T}\chi(0)e^{i2\pi t/T} \mid_{T\to\infty}, \tag{21}$$

respectively. These representations are consistent with the real Grassmannianity of in the limiting sense. In particular, Eq. (21) should be regarded as a "regularized" version of Eq. (11). These might enable one to evaluate the integral of the right-hand side of Eq. (19) over a single but infinite period. Let us discuss this point, carefully. The quantity to be evaluated is

$$I \equiv -i\int_0^T dt\frac{d\chi}{dt}\chi \tag{22}$$

in the limit $T\to\infty$. This is a singular quantity, since the integral, $\int_0^T dt\exp(i4\pi t/T)$, has the structure, $\infty\times 0$. At this juncture, the key factor, $1/T$, appearing on the right-hand side in Eq. (21) should be noticed. In addition, it should be recalled that the Ehrenfest invariant can be defined not only for exact periodic orbits but also for slightly non-periodic orbits: that is, the above integral over a period does not necessarily vanish, in general. Accordingly, we *reinterpret* Eqs. (20) and (21) as $\chi(t) = \chi(0)$ and $d\chi(t)/dt = (2\pi i/T)\chi(0)$, respectively. The former is natural in view of Eq. (11), whereas the latter is consistent with the reality of $d\chi(t)/dt$ only in the limit $T\to\infty$ that is linked with the infinite integral in Eq. (22) in the same limit. In particular, the imaginary unit "i" turns out to be essential in the result to be obtained. Consequently, we have

$$I = 2\pi\chi^2. \tag{23}$$

Thus, the contradiction mentioned above is avoided within the present manipulation. On the other hand, the integral of the left-hand side of Eq. (19) is calculated by the Berezin integration rule in Eq. (16) in conformity with Eq. (4) (recalling that $\chi = \psi_3/\sqrt{2}$ with ψ_3 given in Eq. (4)):

$$-i\int d\chi\chi = \pm\frac{\pi}{2}, \tag{24}$$

leading to the following relation:

$$\chi^2 = \pm\frac{1}{4}. \tag{25}$$

Therefore, the variable

$$c = \sqrt{\pm 4}\chi \tag{26}$$

satisfies

$$c^2 = 1. \tag{27}$$

The evaluation of Eq. (22) discussed above exhibits how Bohr–Sommerfeld quantization of χ is mathematically subtle, primarily concerning the limit $T\to\infty$.

c in Eq. (27) is known as a Clifford number. It is idempotent and anticommutes with Grassmann-odd fermionic quantities. Such a number has been employed, for example, in supersymmetric ensembles at finite temperature [9, 10], and the Wigner phase-space representation of the optical Dicke model [11].

Thus, Bohr–Sommerfeld quantization of the (0+1)-dimensional classical Majorana field may yield a Clifford number. This is the main result of the present work.

To summarize, we have revisited Bohr–Sommerfeld quantization of the fermionic degrees of freedom in order to understand the quantum-classical correspondence for fermions. In particular, we have found that Bohr–Sommerfeld quantization of the Majorana-type variable leads to a Clifford number. However, clearly there still remains a mathematically delicate point behind the logic in evaluating Eq. (22). This reflects an aspect of subtleties in classical theory of fermions, and Bohr-Sommerfeld quantization makes it manifest in a peculiar manner.

Semiclassical theory with Bohr–Sommerfeld quantization is actually not obsolete but of modern interest. It has recently been noticed that the approach of early quantum mechanics can work very efficiently in the limit of infinite dimensions (see [12] and the references therein). We also mention that a field-theoretic generalisation of the present scheme may be useful. In Ref. [13], semiclassical quantisation has been discussed for the Gross–Neveu model [14]. There, a bosonic auxiliary field has been introduced at the classical level in order to make the Lagrangian bilinear in terms of the fermonic fields that have been integrated out to obtain a bosonic field theory to be semiclassically quantized. The scheme discussed in the present work may allow one to quantize the fermonic fields directly without introducing the bosonic auxiliary field.

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