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A note on quantum groups and integrable systems

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Free-field formalism for quantum groups (preprint ITEP-M3/94, CRM-2202 hep-th/9409093) provides a special choice of coordinates on a quantum group. In these coordinates the construction of associated integrable system (arXiv:1207.1869) is especially simple. This choice also fits into general framework of cluster varieties (math.AG/0311245) – natural changes of coordinates are cluster mutations.

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Very often we describe physical systems using perturbation theory. But to develop a perturbation theory, we need some starting point, the model that can be solved exactly – the integrable system. When we encounter new physical system, we need to choose this starting point – and the wider choice we have, the better. Hence, search for integrable systems is very important.

Usually, a system is integrable because it has some symmetry (see e.g. [1–4]). If there is a continuum of symmetries, they form a Lie group. If the system is quantum, then symmetries can form quantum Lie group.

But connection between Lie groups and integrable systems is even more strong. We can actually use a classical Lie group to construct a bunch of integrable systems [5]. The phase spaces of these integrable systems are Poisson submanifolds (leafs) of dimension $2n$, where n is the rank of the group. The integrals of motion are Ad-invariant functions on the group, and there are exactly n of them.

Now one may wonder: *does this construction of integrable systems extend to quantum groups?* This note outlines, how to do this, for the case of quantum group $SL_q(N)$.

The question is harder than it seems. This is because quantum group has a rather technical and indirect definition. One starts with q-deforming the Lie algebra commutational relations in the so-called Chevalley basis. Then one considers all possible functions of these elements. In this space of all functions, one is interested only in group-like elements – functions g, that satisfy equation

$$\Delta(g) = g \otimes g, \tag{1}$$

where Δ is the comultiplication – another important component of a quantum group definition. Solutions of this equation form a group and as $q \to 1$ this set of solutions coincides with classical group $SL(N)$. Thus, it is this set of group-like elements, that is the proper $q \neq 1$ deformation of a Lie group $SL(N)$.

The story is even more tricky than this. If one considers Taylor coefficients to be commutative (e.g. complex numbers), then Eq. (1) does not have interesting solutions. So, one needs to introduce extra $N^2 - 1$ non-commutative parameters, which become commutative in the classical limit. The question is how to introduce these non-commutative variables in such a way, that the result has some physically interesting properties.

A clever approach to this problem, which is simultaneously a better, more direct way to work with group-like variety, is to propose some ansatz for group-like element in terms of new non-commutative parameters [6]. Eq. (1) then would imply some commutational relations for these parameters. There are two crucial features of the Morozov–Vinet ansatz. First, while it uses $N^2 - 1$ parameters, it only uses *simple* roots of the quantum Lie algebra (there are $2N - 2$ of them), but each one multiple times. Second, $q \to 1$ limit of this ansatz looks like exponential map from Lie algebra to its Lie group. As a result, the required commutational relations have very simple form

$$\begin{gathered}\psi_i\psi_j = q^{-C_{[i][j]}}\psi_j\psi_i,\ \chi_i\chi_j = q^{-C_{[i][j]}}\chi_j\chi_i,\ \text{for } i<j,\\ q^{\phi_i}\psi_j = q^{C[i][j]}\psi_j q^{\phi_i},\ q^{\phi_i}\chi_j = q^{C[i][j]}\chi_j q^{\phi_i},\\ \chi_i\psi_j = \chi_j\psi_i,\ \text{for all } i,\ j,\end{gathered} \tag{2}$$

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where $\frac{N(N-1)}{2}$ parameters ψ_i are coupled to negative roots, $\frac{N(N-1)}{2}$ parameters χ_i – to positive roots, and $N-1$ parameters ϕ_i – to Cartan elements, C_{ij} is the $SL_q(N)$ Cartan matrix. The bracket map $[i]$ gives the number of the simple root, associated with the i-th variable. For example, for $SL_q(4)$ we have: $[1]=1$, $[2]=2$, $[3]=1$, $[4]=1$, $[5]=2$, $[6]=1$. These commutational relations are exponents of Heisenberg-like commutational relations, which motivates the name "free-field formalism" – and makes this approach so attractive.

Even though Morozov–Vinet ansatz already leads to simple commutational relations, it is still not clear how to reduce system to $2n$-dimensional subspace. But at $q=1$ this ansatz is very similar to explicit parametrization of classical Lie group element, used in [5] and [7, 8] – with a slight difference – Cartan elements are interspersed with simple roots. For example, in case of $SL(3)$ parametrization of [5] looks like:

$$\begin{gathered} g = H_1(x_0^1)E_1H_1(x_1^1)F_1H_1(x_2^1) \times \\ \times\left(H_2(x_0^2)E_2H_2(x_1^2)F_2H_2(x_2^2)\right)E_1H_1(x_3^1)F_1H_1(x_4^1), \end{gathered} \tag{3}$$

where $E_i=\exp(e_i)$, $F_i=\exp(f_i)$ – exponents of positive and negative simple roots and $H_i(x)=x^{h^i}$ – exponents of Cartan elements. Structurally this ansatz looks like a word of numbers $1\bar{1}2\bar{2}1\bar{1}$ – if i stands for E_i and $\bar{i}$ stands for F_i, with appropriate $H(x)$ inserted between every two numbers (for details, see [8]).

We can use this ansatz as a hint, how to improve Morozov–Vinet ansatz for $q\neq 1$ to make integrable system construction more explicit. The only adjustments, we need to do, is to change exponential functions by appropriate q-exponential functions and (this is a technical detail, related to exact formula for comultiplication), unlike [8], but more like [6], consider $H(x)=x^{h_i}$, where $h_i = 1/2C_{ij}h^j$. Then same group variety equation (1) will lead to even simpler commutational relations

$$\begin{aligned} x_0^1x_1^1 &= q^{-2}x_0^1x_1^1, & x_1^1x_2^1 &= q^2x_2^1x_1^1, \\ x_2^1x_3^1 &= q^{-2}x_3^1x_2^1, & x_3^1x_4^1 &= q^2x_4^1x_3^1, \\ x_0^2x_1^2 &= q^{-2}x_0^2x_1^2, & x_1^2x_2^2 &= q^2x_2^2x_1^2, \end{aligned} \tag{4}$$

and all the other commutators are zero. This can be conveniently encoded in directed graphs

$$\begin{tikzcd} x_0^1 \arrow[r] & x_1^1 & x_2^1 \arrow[l] \arrow[r] & x_3^1 & x_4^1 \arrow[l] \\ x_0^2 \arrow[r] & x_1^2 & x_2^2 \arrow[l] \end{tikzcd} \tag{5}$$

which generalizes to $SL_q(N)$ case in an obvious way (for $SL_q(4)$ there will be 3 graphs with 3, 5, and 7 vertices, respectively). We see that variables split into $N-1$ groups, which do not interact with each other. It is straightforward to see (taking into account difference between h_i and h^i) that these commutational relations reproduce Poisson bracket of [8] in $q\to 1$ limit.

In these variables it is easy to see how to do reduction to $2n$-leaf. It is clear, that if we consider only functions that depend (in $SL_q(3)$ case) on x_0^1, x_1^1, x_0^2, and x_1^2 (i.e. on first two coordinates coupled to each simple root, for general N), the space of such functions is closed under commutational relations (4). This means, that it defines a $2n$-dimensional submanifold in quantum group $SL_q(N)$. Points of this manifold can be explicitly parametrized as

$$g_{2n} = \prod_{i=1}^{N-1}(x_0^i)^{h_i}\mathcal{E}_q(e_i)(x_1^i)^{h_i}\mathcal{E}_{1/q}(f_i), \tag{6}$$

where $\mathcal{E}_q(x)$ is the q-exponential. Taking the limit $q\to 1$ in this expression, we obtain formula for $2n$ symplectic manifold of a classical Lie group from [9], on which a relativistic Toda chain lives. Thus, it is natural to think, that g_{2n} is the phase space of q-deformed relativistic Toda chain.

Finally, looking at commutational relations (4) we see, that the structure of cluster variety, which is present at $q=1$ survives quantization and is promoted to structure of quantum cluster variety [7]. Actually, if, for example, we change the order of first positive and first negative root in the quantum ansatz, so that it becomes

$$\begin{gathered} g = H_1(y_0^1)F_1H_1(y_1^1)E_1H_1(y_2^1) \times \\ \times\left(H_2(x_0^2)E_2H_2(x_1^2)F_2H_2(x_2^2)\right)E_1H_1(x_3^1)F_1H_1(x_4^1), \end{gathered} \tag{7}$$

where $H_i(x)=x^{h_i}$, $E_i=\mathcal{E}_q(e_i)$, and $F_i=\mathcal{E}_{1/q}(f_i)$, we would get that this new parametrization is related to the old one by simple non-commutative change of variables

$$y_0^1 = x_0^1(1+qx_1^1),\ y_2^1 = x_2^1(1+qx_1^1),\ y_1^1 = 1/x_1^1. \tag{8}$$

Changes of variables of this form are precisely cluster mutations and are, in fact, also encoded in the picture (5).

To summarize, even though there are a lot of peculiarities and details that need to be kept in mind when doing practical calculations, the general picture is clear and simple. Construction of integrable systems of [5] generalizes to $q\neq 1$ case almost literally. There it no longer looks *ad hoc* – all the choices are fixed by group variety equation (1), an essential component of the quantum group. Furthermore, the rich structure of cluster variety is fully present in the quantum case – commutational relations and natural changes of coordinates (mutations) are encoded in cluster quivers like

(5). Even though we performed our calculations only for $SL_q(N)$ group we hope everything is true for other Lie groups – but this is subject of future research.

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