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Adaptive Quantum Tomography

S. S. Straupe[1)]

Faculty of Physics, Lomonosov Moscow State University, 119001 Moscow, Russia

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Quantum state tomography is a general notion standing for a set of statistical methods for the reconstruction of a density matrix, describing an unknown quantum state, using the experimental data. The first solution to the problem was developed by Hradil back in 1997 [1], and it was based on the well-known concept of maximum-likelihood estimation (MLE). A different perspective on quantum state estimation is offered by Bayesian statistics [2, 3]. Here inference is based on consistent application of the Bayes rule for conditional probability. At first some *prior* probability distribution over the state space $p(\rho)$ should be specified. When the measurements are performed and the data collected, the distribution should be updated using the Bayes rule to obtain the *posterior* distribution $p(\rho|\mathcal{D}) \propto \mathcal{L}(\rho;\mathcal{D})p(\rho)$, where $\mathcal{D} = \{\gamma_j\}$ is the set of all measurement outcomes γ_j – the actual data, used for inference. A natural point estimate for the unknown state would be a *Bayesian mean estimate* (BME) $\hat{\rho}_{BME} = \int \rho p(\rho|\mathcal{D})d\rho$. In contrast to MLE, besides a point estimate, Bayesian inference provides a whole distribution, which may be used to estimate error bars, as well as to obtain estimates for any properties of the state of interest as averages over the posterior. We are particularly interested in the Bayesian approach to state estimation, because it allows for an easy implementation of adaptive measurement strategies.

In order to quantify the deviation of the state from some ideal one, which the experimentalist intended to prepare, we need to define the distance on the space of quantum states. The most widely used distinguishability measure for the tomographic scenario is the *fidelity*: $F(\rho_1,\rho_2) = \left[\mathrm{Tr}\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right]^2$.

Different *tomography protocols* – estimation strategies, based on a particular choice of an estimator and measurements, should be compared to each other and optimized for precision. We are mostly concerned with the behavior of infidelity as a function of the *sample size* N – the overall number of measurement outcomes registered. The ultimate limits of precision are known at least for qubits, and are set by collective protocols, where a complicated measurement is performed on the whole ensemble of N qubits. The problem was considered for pure states in [4] where the bound for fidelity, now known as the *Massar–Popescu bound* was derived: $F \leq (N+1)/(N+2)$. At the same time tomography with local measurements achieves $1 - F \sim \sqrt{N}$ only. A natural question arises: is it possible to come closer to this ultimate bound with local (one qubit at a time) measurements? The answer turns out to be positive, but the structure of the protocol should be changed significantly – it should become an adaptive estimation scheme.

One can show, that if the state was known a priori, the best strategy would have been to make measurements in the basis, which diagonalizes ρ [5]. Now it is straightforward to come up with a simple adaptive strategy: one has to perform standard 6-state tomography on some fraction of N_0 qubits and obtain a preliminary estimate for ρ. Then one rotates the measurement basis, such that it coincides with the eigenbasis of ρ and performs the remaining $N-N_0$ measurements. The first experimental demonstration of two-step adaptive tomography was performed for polarization qubits and confirmed the quadratic improvement in infidelity scaling [5].

A natural extension of the two-step adaptive strategy is a fully adaptive protocol, where the measurement basis is realigned with the current estimate after every measurement. Such a protocol was considered in [6] for a 4-state optimal POVM. However, to the best of our knowledge, it was never implemented experimentally. This kind of adaptive strategy is a particular case of a general approach, known as *self-learning measurements* or *adaptive experimental design*. Let $\mathbb{M}_\alpha$ be the POVM, corresponding to some particular choice of settings in the experimental apparatus, which we denote α. From the Bayesian point of view an algorithm for self-learning measurements may be reduced to the following scheme [7]: the first measurement setting α_0 is chosen at random; n-th measurement is chosen by optimizing a selected *utility function* $U(\alpha,\mathcal{D})$ averaged over the possible measurement outcomes: $\alpha_n = \arg\max_\alpha \sum_{\gamma_n} p(\gamma_n|\alpha)U(\alpha,\mathcal{D}_n)$. Here the probability $p(\gamma_n|\alpha)$ of observing a particular outcome γ_n should be calculated using the current posterior distribution for the state ρ: $p(\gamma_n|\alpha) = \int d\rho p(\rho|\mathcal{D}_{n-1})\mathrm{Tr}\left[M_{\alpha\gamma_n}\rho\right]$.

[1)]e-mail: straups@yandex.ru

Utility functions may be roughly divided in two groups. In the first one the utility functions are constructed to optimize the parameters of the estimate. Examples are A-optimality, where the minimized quantity is the trace of the covariance matrix for the estimate and D-optimality, where the determinant of the same matrix is minimized. The second group focuses on the *information gain* of an experiment. On information-theoretic grounds the utility function here is chosen to be the expected relative entropy between the posterior $p(\rho|\mathcal{D},\alpha)$ and the prior $p(\rho)$ distributions: $U(\alpha,\mathcal{D}) = \int d\rho p(\rho|\mathcal{D},\alpha)\log\frac{p(\rho|\mathcal{D},\alpha)}{p(\rho)}$. Both types of utility functions were used in the design of adaptive protocols for quantum tomography. Adaptive protocols demonstrate the optimal scaling of infidelity $1-F = aN^{-1}$, with the prefactor a slightly varying depending on the particular choice of the utility function and other details of the protocol.

Adaptive Bayesian experimental design for optimal quantum tomography of qubits was first proposed in [8] and performed experimentally by Hannemann et al. [9]. The optimal measurements were precomputed and stored in the look-up table, rather than be performed online. Online implementation of self-learning measurements was first demonstrated in [10]. The first experiment, where full tomography of qubits with self-learning measurements was implemented online in the course of experiment, was reported by Kravtsov et al. [11]. This was made possible by the development of a fast Bayesian protocol using the information gain as a utility function by Huszár and Houlsby [12]. The experimental results of [11] confirmed the advantage of the adaptive protocol over both standard 6-state MUB measurements and random measurements, sampled from the optimal uniform POVM.

Implementing adaptive strategies for multi-qubit states is hard for two main reasons. First of all, the computational complexity of high-dimensional optimization required to find an optimal measurement quickly becomes overwhelming for most of the self-learning strategies. Second problem is that optimal measurements in high dimensions almost certainly turn out to be projections on entangled states, which are extremely hard to do in experiment. There may be, however, paths to partly overcome both problems, which became recently an area of active research. The first route, recently taken in [13] and [14], is based on using optimization algorithms, that are fast enough to be implemented online even for systems of several qubits. Both works report experiments with polarization states of correlated photon pairs.

There are no experimental demonstrations going beyond two qubits so far, and that may be challenging in terms of computational time even for the fastest algorithms. Algorithms, based on heuristics, rather than precise evaluation of the utility function, with self-guided tomography [15, 16] being an example, may be fast enough to be used online in experiments with few qubits. It would be interesting to see such experiments performed in the nearest future.

The full text of this review article will the full list of references and figures be published in JETP Letters. In the full article we discuss several statistical frameworks for adaptive experimental design. We argue in favor of the Bayesian approach, highlighting both its advantages for a statistical reconstruction of unknown quantum states and processes, and utility for adaptive experimental design. The discussion is supported by an analysis of several recent experimental implementations and numerical recipes.

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