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On Hopf-induced Deformation of Topological Locus

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Hopf link invariant is the Wilson average in the gaige Chern–Simons theory. The invariant plays a key role in many modern branches of physics, from topological strings to quantum computers. Since the theory is topological, the Wilson average depends only on the gauge coupling constant κ encoded in the variable $q = \exp(4\pi i/\kappa)$, on the gauge group and on the representation. We consider the gauge group $SU(N)$ (HOMFLY invariant) and two generally distinct representations associated with two components of the link.

There are three ways to represent answers for this Wilson average in arbitrary representations. The first way is the Rosso–Jones formula

$$\mathcal{H}^{\text{Hopf}}_{R\times S} = q^{\varkappa_R+\varkappa_S} \sum_{Q\in R\otimes S} N^Q_{RS} \cdot q^{-\varkappa_Q} \cdot D_Q(q), \qquad (1)$$

where N^Q_{RS} are integer-valued Littlewood–Richardson coefficients,

$$\varkappa_Q = (\Lambda_Q, \Lambda_Q + 2\rho) \qquad (2)$$

are the eigenvalues of the second Casimir operator,

$$D_Q = \prod_{\alpha\in\Delta_+} \frac{[(\Lambda_Q+\rho,\alpha)]}{[(\rho,\alpha)]} \qquad (3)$$

is the quantum dimension. Λ_Q is the highest weight in representation Q, ρ is the Weyl vector equal to the half sum of positive roots, and square bracket denotes the quantum number:

$$[x] = \frac{q^x - q^{-x}}{q-q^{-1}} = \frac{\{q^x\}}{\{q\}}, \qquad \{x\} = x - x^{-1}. \qquad (4)$$

For the group $SU(N)$, the whole N-dependence is encoded in the parameter $A = q^N$, for exception of the $U(1)$-factor $q^{\frac{2|R||S|}{N}}$, the Wilson average being a rational function of q and A with a simple denominator. In this case, for the representation R the second Casimir is equal to

$$\varkappa_R = 2\kappa_R - \frac{|R|^2}{N} + |R|N \qquad (5)$$

with $\kappa_R = \sum_{r_{i,j}\in R}(j-i)$, where sum runs over the boxes of the Young diagram R.

The second form of answers for the Hopf link invariant are the very explicit finite sums. This realization of the Hopf polynomials is most convenient in concrete calculations, but at the moment Hopf invariants are available in this form not in the most general case.

At last, the third kind of explicit formulas are the formulas of the form

$$\mathcal{H}^{\text{Hopf}}_{S\times R} = D_S \cdot \text{Schur}_R\{p^{*S}\}. \qquad (6)$$

The Schur functions $\text{Schur}_R\{p^{*S}\}$ are evaluated here at special points p^{*S}, which can be considered as deformations of the topological locus

$$p^*_k = \frac{[Nk]}{[k]} = \frac{\{A^k\}}{\{q^k\}}, \qquad (7)$$

which emerges in quantum dimensions: $D_S = \text{Schur}_S\{p^*\}$. The explicit formula for these special points is

$$p^{*R}_k = q^{\frac{2|R|k}{N}} \Big(p^*_k + A^{-k} \sum_i q^{(2i-1)k}(q^{-2kr_i}-1)\Big), \qquad (8)$$

where the factor is necessary for the correct accounting of the $U(1)$-factor in the Wilson average.

In the case of most general finite-dimensional representations called *composite* (or rational) given by a pair of the Young diagrams (R,P), the deformed topological locus is of the form

$$p^{*(R,P)}_k = q^{2\frac{|R|-|P|}{N}k} \times \\ \times \Big(p^*_k + \frac{1}{A^k} \cdot \sum_{j=1}^{l_R} q^{(2j-1)k} \cdot (q^{-2kr_j}-1) + \\ + A^k \cdot \sum_{i=1}^{l_P} q^{(1-2i)k} \cdot (q^{2kp_i}-1)\Big). \qquad (9)$$

The corresponding Schur functions are, in this case,

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$$\mathrm{Schur}_{(R,P)}\{p^{*S}\} = \sum_{\eta\in R\cap P^{\mathrm{tr}}} (-)^{|\eta|}\cdot \mathrm{Schur}_{R/\eta}\{p^{*S}\}\times \\ \times\, \mathrm{Schur}_{P/\eta^{\mathrm{tr}}}\{p^{*S}(A^{-1},q^{-1})\}.$$

Thus, the Wilson average in the most general case of two composite representations is

$$\mathcal{H}^{\mathrm{Hopf}}_{(R,P)\times(S,Q)} = D_{(R,P)}\cdot \mathrm{Schur}_{(S,Q)}\{p^{*(R,P)}\} = \\ = D_{(S,Q)}\cdot \mathrm{Schur}_{(R,P)}\{p^{*(S,Q)}\}.$$

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