Pis'ma v ZhETF, vol. 108, iss. 10, pp. 721 – 722

New symmetries for the $U_q(sl_N)$ 6-j symbols from the Eigenvalue conjecture

A. Morozov[+*×], A. Sleptsov[+*×○1)]

[+]Moscow Institute of Physics and Technology, 141701 Dolgoprudny, Russia

[*]Institute for Information Transmission Problems, 127994 Moscow, Russia

[×]Institute for Theoretical and Experimental Physics, 117218 Moscow, Russia

[○]Laboratory of Quantum Topology, Chelyabinsk State University, 454001 Chelyabinsk, Russia

Submitted 4 Octobert 2018

DOI: 10.1134/S0370274X18220101

Racah coefficients (6-j symbols) are an important quantity in the theoretical and mathematical physics. They appear everywhere from quantum mechanics to the knot theory and integrable systems. Despite this there are a lot of open problems with Racah coefficients even for $U_q(sl_N)$. In the case of $U_q(sl_2)$ there is a general formula for Racah coefficients derived by A. Kirillov and N. Reshetikhin. According to that formula 6-j symbols are described by the q-hypergeometric function ${}_4\Phi_3$. Using this fact the whole set of symmetries of $U_q(sl_2)$ Racah coefficients was described.

The next step was done by extending the explicit Racah formula to the symmetric representations of arbitrary $U_q(sl_N)$. It covers only so-called *exclusive* 6-j symbols, which appear in the arborecent links calculus coming from the Wess–Zumino–Novikov–Witten Confromal Field Theory (WZNW CFT) consideration.

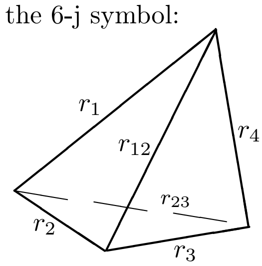
The modern version of the Reshetikhin–Turaev approach provide a general method to compute colored HOMFLY polynomials for arbitrary link. Racah coefficients play the key role in this formalism, because they relate different $\mathcal{R}$-matrices. Some of the Racah coefficients appear in the Yang–Baxter equation together with $\mathcal{R}$-matrices. This led to the eigenvalue conjecture which relates the $\mathcal{R}$-matrix eigenvalues with the Racah coefficients. An eigenvalue conjecture is a very important concept which has a very useful practical applications. This conjecture states that Racah coefficients are fully defined by the set of normalised eigenvalues of the corresponding $\mathcal{R}$-matrices. In its strong form the exact form for Racah coefficients from the $\mathcal{R}$-matrix eigenvalues is provided, and it is known for the matrices of the size less or equal to 5×5. In its weak form it allows for example to calculate knot polynomials in any symmetric representation, expressing all the needed Racah coefficients through the known $U_q(sl_2)$ Racah coefficients. The eigenvalue conjecture provides a certain symmetry for the Racah coefficients.

The eigenvalue conjecture gives rise to the question what happens with this symmetry in the well-known $U_q(sl_2)$ case. In the present paper we prove that: **for $U_q(sl_2)$ Racah coefficients the eigenvalue conjecture is provided by the Regge symmetry.**

However, unlike the Regge symmetry, the eigenvalue conjecture is also formulated for arbitrary rank $U_q(sl_N)$. Thus one can hope that studies of the eigenvalue conjecture can give us some insights into the generalization of known symmetries from $U_q(sl_2)$ to $U_q(sl_N)$ case.

In the case of $U_q(sl_2)$ all symmetries of 6-j symbols are known due to their representation in terms of q-hypergeometric function. The symmetry group contains 144 elements, the full tetrahedral group S_4 is its subgroup. It describes *the tetrahedral* symmetry, which can be represented in a pictorial way by the associating a corresponding tetrahedron with the 6-j symbol:

$$\left\{\begin{matrix} r_1 & r_2 & r_{12} \\ r_3 & r_4 & r_{23} \end{matrix}\right\} =$$



Other symmetries are given by the q-analog of *the Regge* symmetry. Let $p=\frac{1}{2}(r_1+r_2+r_3+r_4)$, then the Regge symmetry holds:

$$\left\{\begin{matrix} r_1 & r_2 & r_{12} \\ r_3 & r_4 & r_{23} \end{matrix}\right\} = \left\{\begin{matrix} p-r_1 & p-r_2 & r_{12} \\ p-r_3 & p-r_4 & r_{23} \end{matrix}\right\}. \quad (1)$$

In the case of $U_q(sl_N)$ we know only about the tetrahedral symmetry. The counterpart of the Regge symmetry is still unknown.

1) e-mail: sleptsov@itep.ru

We prove that the eigenvalue conjecture is true if $R_1 = R_2 = R_3 = [r]$ for $U_q(sl_2)$. Moreover, we show that it follows from the Regge symmetry. Indeed, in this case the eigenvalue conjecture predicts that there is only one pair of non-identical Racah matrices of the size $d\times d$ satisfying the conditions of the conjecture whose matrix elements (6-j symbols) can be written as follows:

$$\boxed{\left\{\begin{array}{ccc} r & r & 2r-2k_1 \\ r & 3r-2d+2 & 2r-2k_2 \end{array}\right\}, \quad k_1,k_2 = 0\dots d-1} \tag{2}$$

$$\boxed{\left\{\begin{array}{ccc} 2r-d+1 & 2r-d+1 & 2r-2k_1 \\ 2r-d+1 & d-1 & 2r-2k_2 \end{array}\right\}, \quad k_1,k_2 = 0\dots d-1} \tag{3}$$

Thus, the eigenvalue conjecture predicts that 6-j symbol (2) coincides with (3). With the help of the Regge symmetry we can easily check that it is true.

In the same way the eigenvalue conjecture predicts the symmetry between different Racah matrices for the quantum group of the rank higher than 2. Such cases are more involved because they usually contain multiplicities. There is no analytical description of 6-j symbols like for $U_q(sl_2)$, consequently we can only perform some checks of the eigenvalue conjecture.

• In the papers [1, 2] there were found general nontrivial solutions of the quantum Yang–Baxter equation if the size of the R-matrix not greater than 5 and all its eigenvalues are different. These solutions provide explicit formulas for matrix elements (i.e., for 6-j symbols) in terms of algebraic functions depending on eigenvalues. Therefore, these formulas provide the proof of the eigenvalue conjecture for this particular class of matrices. Note these formulas are valid for any rank N of the algebra $U_q(sl_N)$.

• If the size of the Racah matrix is greater than 5, then there are no explicit solutions of the Yang–Baxter equation. However we manage to present some particular examples of the Racah matrices, which confirm the eigenvalue conjecture. These examples we computed in our previous papers:

$$U\left[\begin{array}{cc} [3,3] & [3,3] \\ [3,3] & [7,5,3,3] \end{array}\right] = U\left[\begin{array}{cc} [3,3] & [3,3] \\ [3,3] & [6,6,4,2] \end{array}\right], \qquad 6\times 6,\ \text{multiplicity free} \tag{4}$$

$$U\left[\begin{array}{cc} [3,3] & [3,3] \\ [3,3] & [8,5,3,2] \end{array}\right] = U\left[\begin{array}{cc} [3,3] & [3,3] \\ [3,3] & [7,6,4,1] \end{array}\right], \qquad 6\times 6,\ \text{double multiplicity} \tag{5}$$

$$U\left[\begin{array}{cc} [4,2] & [4,2] \\ [4,2] & [5,5,3,3,2] \end{array}\right] = U\left[\begin{array}{cc} [4,2] & [4,2] \\ [4,2] & [6,4,3,3,1] \end{array}\right], \qquad 9\times 9,\ \text{two double multiplicities.} \tag{6}$$

Even these few examples give us a hope that further studies of the eigenvalue conjecture will provide some general symmetry of the Racah coefficients for any $U_q(sl_N)$ group. However at the moment some general expressions of the eigenvalue conjecture are lacking and this remains to be done. Based on the sl_2 consideration we can provide the $U_q(sl_N)$ version of the same relation. The formulae (2) and (3) become:

$$\begin{array}{c}\left[\begin{array}{ccc} [2r-N+1] & [2r-N+1] & [3r-N+1-k_1, k_1+r-N+1] \\ [2r-N+1] & [3r-N+1, 3r-2N+2] & [3r-N+1-k_2, k_2+r-N+1] \end{array}\right] = \\ = \left[\begin{array}{ccc} [r] & [r] & [2r-k_1, k_1] \\ [r] & [3r-N+1, N-1] & [2r-k_2, k_2] \end{array}\right], \quad k_1,k_2 = 0\dots N-1 \end{array}. \tag{7}$$

We hope that this formula is just the beginning of the formulation of Racah coefficients symmetries in the $U_q(sl_N)$ case.

Full text of the paper is published in JETP Letters journal. DOI: 10.1134/S0021364018220058

1. H. Itoyama, A. Mironov, A. Morozov, and An. Morozov, Int. J. Mod. Phys. A **28**, 1340009 (2013); arXiv:1209.6304.

2. I. Tuba and H. Wenzl, Pacific Journal of Mathematics **197**(2), 491 (2001); arXiv:math/9912013.