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Analog of gravitational anomaly in topological chiral superconductors

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Topological materials with Weyl fermions provide the possibility to study quantum anomalies, such as the Adler–Bell–Jackiw (ABJ) chiral anomaly [1, 2]. The condensed matter analog of the ABJ anomaly has been experimentally probed in the Weyl superfluid – the chiral A-phase of liquid ^{3}He [3]. In this electrically neutral matter, the chiral anomaly is caused by the effective electromagnetic field produced by deformations. The analogs of gravitational anomalies have been also discussed. In particular, the analog of the Nieh–Yan anomaly in terms of torsion [4–6] has been studied [7–13]. Recent discussion of anomalies in Weyl materials can be found in [14]. Here we consider the $p+ip$ superconductors and show that the $U(1)$ electromagnetic field plays the role of spin connection in the effective tetrad gravity. These superconductors experience the gravitational anomaly. As distinct from the conventional chiral anomaly, the corresponding ABJ equation contains extra factor 1/3.

The Hamiltonian for Bogoliubov quasiparticles in the $p+ip$ superfluid ^{3}He-A has the following form [15]:

$$H(\mathbf{p}) = \begin{pmatrix} \frac{p^2}{2m} - \mu_0 & \mathbf{e}_1\cdot\mathbf{p} + i\mathbf{e}_2\cdot\mathbf{p} \\ \mathbf{e}_1\cdot\mathbf{p} - i\mathbf{e}_2\cdot\mathbf{p} & -\frac{p^2}{2m} + \mu_0 \end{pmatrix}. \quad (1)$$

Here $\mathbf{e}_1$ and $\mathbf{e}_2$ are the real vectors satisfying conditions $\mathbf{e}_1\cdot\mathbf{e}_2 = 0$, $\mathbf{e}_1^2 = \mathbf{e}_2^2$; $\mu_0 = p_0^2/2m$. This Hamiltonian contains two Weyl points – topologically stable nodes in the energy spectrum, which represent monopoles in the Berry phase in momentum space [16] (classification in terms of two topological invariants in the interacting systems can be found in [17]). The nodes are at $\mathbf{p}_\pm = \pm p_0\hat{\mathbf{l}}$, where $\hat{\mathbf{l}} \parallel \mathbf{e}_1\times\mathbf{e}_2$. Deformations produce the effective electromagnetic field $\mathbf{A}^{\rm eff}(\mathbf{r},t) = p_0\hat{\mathbf{l}}(\mathbf{r},t)$, or the pseudo-magnetic field in terminology of [18], giving rise to the observed analog of chiral anomaly [3]:

$$\partial_\mu J_5^\mu = \frac{1}{32\pi^2} e^{\mu\nu\rho\sigma} F^{\rm eff}_{\mu\nu} F^{\rm eff}_{\rho\sigma}. \quad (2)$$

Let us now consider the $p+ip$ superconductor and the role of the electromagnetic field in the chiral anomaly neglecting the effective electromagnetic field, $F^{\rm eff}_{\mu\nu} = 0$. The inverse Green's function is

$$G^{-1} = -i\partial_t + \tau_3 q A_0(\mathbf{r},t) + H(\mathbf{p} + \tau_3 q\mathbf{A}(\mathbf{r},t)), \quad (3)$$

where A_μ is the vector potential of electromagnetic field, which acts in different way on particles and holes; and $q=-1$ is electric charge.

Since $\tau_3 = \frac{1}{2i}(\tau_1\tau_2 - \tau_2\tau_1)$, the Green's function can be rewritten in terms of the "covariant derivative":

$$D_i = \partial_i + \frac{1}{8} C_i^{ab}(\tau_a\tau_b - \tau_b\tau_a), \quad D_t = \partial_t + \frac{1}{8} C_0^{ab}(\tau_a\tau_b - \tau_b\tau_a), \quad (4)$$

see, e.g., [10]. Components of spin connection are expressed in terms of the electromagnetic field:

$$C_i^{12} = -C_i^{21} = 2A_i(\mathbf{r},t), \quad C_0^{12} = -C_0^{21} = 2A_0(\mathbf{r},t). \quad (5)$$

In zero effective gauge field, $F^{\rm eff}_{\mu\nu} = 0$, the chiral anomaly comes only from the curvature of the effective gravitational field. For a single Weyl node one has [19]:

$$\partial_\mu J_5^\mu = \frac{1}{768\pi^2} e^{\mu\nu\rho\sigma} R^{ab}_{\mu\nu} R^{cd}_{\rho\sigma}\eta_{ad}\eta_{bc}, \quad (6)$$

where the curvature is:

$$R^{ab}_{\mu\nu} = \nabla_\mu C^{ab}_\nu - \nabla_\nu C^{ab}_\mu + (C^{ac}_\mu C^{db}_\nu - C^{ac}_\nu C^{db}_\mu)\eta_{cd}. \quad (7)$$

According to Eq. (5) one obtains:

$$\partial_\mu J_5^\mu = \frac{1}{384\pi^2} e^{\mu\nu\rho\sigma} R^{12}_{\mu\nu} R^{21}_{\rho\sigma} = \frac{1}{96\pi^2} e^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}, \quad (8)$$

$$\partial_\mu J_5^\mu = \frac{1}{24\pi^2}\mathbf{E}\cdot\mathbf{B} = \frac{1}{3}\frac{1}{8\pi^2}\mathbf{E}\cdot\mathbf{B}. \quad (9)$$

The gravitational anomaly becomes the gauge anomaly, with the extra factor 1/3 compared with ABJ equation (2) for the effective field $\mathbf{A}^{\rm eff} = p_0\hat{\mathbf{l}}$. This may have relation to the consistent anomaly [18], and also to the factor 1/3 obtained in [20] for the electromagnetic response and θ-term in the gapped topological superconductors [21–23].

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In the electrically neutral superfluids the vector potential $\mathbf{A}$ is substituted by superfluid velocity: $\mathbf{A} \to \to m\mathbf{v}_s$, and the gravitational anomaly becomes

$$\partial_\mu J_5^\mu = \frac{m^2}{24\pi^2}\dot{\mathbf{v}}_s \cdot (\nabla \times \mathbf{v}_s). \quad (10)$$

The anomaly is produced by the instanton – the process of creation of the 3D skyrmions – hopfions in $\hat{\mathbf{l}}$-field [24]. In ^{3}He-A, the density of the hopfion topological charge and its current are expressed in terms of $\mathbf{v}_s$:

$$n_H^0 = \frac{m^2}{4\pi^2}(\mathbf{v}_s \cdot (\nabla \times \mathbf{v}_s)), \; N_H = \int d^3r\, n_H^0, \quad (11)$$

$$\mathbf{n}_H = \frac{m^2}{4\pi^2}(\mathbf{v}_s \times \partial_t \mathbf{v}_s)), \quad (12)$$

with the following (non)conservation law:

$$\partial_\mu n_H^\mu = \frac{m^2}{2\pi^2}(\partial_t \mathbf{v}_s \cdot (\nabla \times \mathbf{v}_s)). \quad (13)$$

The process of the change of the topological charge is the π_3 instanton: $\partial_\mu n_H^\mu = \delta(t)\delta(\mathbf{r})$. In high energy physics this is the gravitational instanton [25, 26]. So we have

$$\partial_\mu J_5^\mu = \frac{1}{6}\partial_\mu n_H^\mu, \quad (14)$$

i.e. the gravitational instanton is accompanied by creation of 6 chiral fermions – the gravitational analog of the Kuzmin–Rubakov–Shaposhnikov scenario of the anomalous electroweak baryogenesis [27].

The Hamiltonian in [28] for Weyl points with topological charges N and $-N$ is:

$$H(\mathbf{p}) = \begin{pmatrix} \frac{p^2}{2m} - \mu_0 & (\mathbf{e}_1 \cdot \mathbf{p} + i\mathbf{e}_2 \cdot \mathbf{p})^N \\ (\mathbf{e}_1 \cdot \mathbf{p} - i\mathbf{e}_2 \cdot \mathbf{p})^N & -\frac{p^2}{2m} + \mu_0 \end{pmatrix}. \quad (15)$$

The corresponding spectrum near the Weyl point is

$$E^2(\tilde{\mathbf{p}} = \mathbf{p} \mp p_0\hat{\mathbf{l}}) = (g^{\perp ik}\tilde{p}_i\tilde{p}_k)^N + g^{\parallel ik}\tilde{p}_i\tilde{p}_k, \quad (16)$$

$$g^{\perp ik} = e_1^i e_1^k + e_2^i e_2^k = g^{ik} - g^{\parallel ik}\;,\; g^{\parallel ik} = e_3^i e_3^k. \quad (17)$$

This represents the anisotropic extension of Hořava gravity [29], see also [30]. The gravitational anomaly leads to $U(1)$ anomaly, again with extra factor 1/3:

$$\partial_\mu J_5^\mu = \frac{N}{96\pi^2} e^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}. \quad (18)$$

We demonstrated that in the chiral superconductors with Weyl fermions the external electromagnetic field serves as the spin connection leading to the gravitational anomaly, which is described by the ABJ equation with an extra factor 1/3 compared with the ABJ equation for the conventional chiral anomaly. In neutral chiral superfluids this gravitational anomaly leads to the gravitational analog of the Kuzmin–Rubakov–Shaposhnikov electroweak baryogenesis.

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