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FROM VIRASORO CONSTRAINTS IN KONTSEVICH'S MODEL TO W-CONSTRAINTS IN ASYMMETRIC 2-MATRIX MODEL

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The Ward identities in Kontsevich-like 1-matrix models are used to prove at the level of discrete matrix models the suggestion of Gava and Narain, which relates the degree of potential in 2-matrix model to the form of $\mathcal{W}$-constraints imposed on its partition function.

1. Matrix models, originally developed [1] as an alternative approach to (so far 2-dimensional) quantum gravity, are now understood to possess a deep and interesting mathematical structure of their own. Indeed, their partition functions are usually subjected to infinite sets of constraints, which can be formulated in a form of differential equations with respect to the "time"-variables, and as a corollary they appear proportional to "τ-functions" of integrable hierarchies [2].

While this general scheme is more or less well understood at the level of discrete 1-matrix models [3,4], it is still in many respects an open question when continuum limit and/or *multi*matrix models are concerned. An important breakthrough in the study of continuum limit of 1-matrix models is due to a recent conjecture of M.Kontsevich [5], who in fact suggested that the proper continuum limit of the Hermitean 1-matrix model can be described in terms of a somewhat different matrix model, to be referred later as Kontsevich's model. Its partition function is essentially proportional to the $N \times N$ (anti)Hermitean matrix integral

$$\mathcal{F}\{\Lambda\} \equiv \int DX \ \exp(-\mathrm{tr}W[X] + \mathrm{tr}\Lambda X) \tag{1}$$

with $W[X] = X^3$ and $N \longrightarrow \infty$. This conjecture was strongly supported by a recent proof [6] that partition function of Kontsevich's model satisfies exactly the same set of differential equations (Virasoro constraints [2]) as continuum limit of the original Hermitean 1-matrix model (so that it remains only to understand what are the requirements, which guarantee the uniqueness of the solution of these Virasoro constraints). In [1] it was shown that Virasoro constraints arise from the obvious Ward identities, satisfied by $\mathcal{F}(\Lambda)$ from (1),

$$(\mathrm{tr}\Lambda^p W'[\partial/\partial\Lambda_{\mathrm{tr}}] + \mathrm{tr}\Lambda^{p+1})\mathcal{F}\{\Lambda\} = 0 \tag{2}$$

as a result of a change (Miwa transformation) of the argument

$$\Lambda \rightarrow \{T_m = \frac{1}{m+\frac{1}{2}}\mathrm{tr}\Lambda^{-m-\frac{1}{2}} + \frac{4}{3\sqrt{3}}\delta_{m,1}\}. \tag{3}$$

In its turn, eq.(2) states that the integral (1) is invariant under infinitesimal shift of integration variable,

$$X \rightarrow X + \Lambda^p. \tag{4}$$

The implication of Kontsevich's conjecture in this context (for more physical implication see ref.[6]) is that the Miwa transformation (2) allows us to substitute a sophisticated double-scaling limit in conventional matrix models [1] by a naive limit of $N \longrightarrow \infty$ in the model like (1). This opens promising possibilities in the study of continuum matrix models.

Somewhat unexpectedly this fresh view on continuum limits provides also a new approach to the study of *discrete* multimatrix models. One of the main problems about them is the lack of understanding (and even a derivation) of differential equations which substitute Virasoro constraints of 1-matrix models (and are believed [2] to be expressible in terms of generators of $\mathcal{W}$-algebras). What we are going to demonstrate in this letter is that just the same Ward identities (2) for the matrix integral (1) provide a simple proof of the $\mathcal{W}$-like constraints in a discrete (i.e. at finite N) 2-matrix model.

Moreover, the recent observation due to E.Gava and K.Narain [7], stating that the spin of $\mathcal{W}$-constraint in fact coincides with the power of potential $W[X]$ is naturally explained in this way. (This suggestive result has been obtained in [7] by a tedious examination of continuum limit of loop equations for specific 2-matrix model with $W[X] = X^3$).

2. The partition function of the discrete Hermitean 2-matrix model [8] is given by a double integral over $N \times N$ Hermitean matrices X and Λ :

$$Z_{V,W} = \int DX\, D\Lambda \, \exp(-\mathrm{tr}\{V[\Lambda] + W[X] + \Lambda X\}) \equiv \int D\Lambda\, e^{-\mathrm{tr}V[\Lambda]} \mathcal{F}_W\{\Lambda\}. \tag{5}$$

The potentials V and W are conventionally parameterized by the corresponding time-variables

$$V[\Lambda] = \sum_{k\geq 0}^{\infty} t_k \Lambda^k, \quad W[X] = \sum_{k\geq 0}^{K} s_k X^k \tag{6}$$

and the partition function $Z_{V,W}$ is usually treated as a functional of $\{t_k\}$ and $\{s_k\}$. Below we will use the obvious notation $Z_{V,W} \equiv Z_W\{t_k\}$. While in [2] it was suggested that the continuum (*i.e.* $N \rightarrow \infty$) limit of $Z_{V,W}$ in the case of $W = V$ (this $K = \infty$) is annihilated by a set of operators which form $\mathcal{W}_3$-algebra, the results of ref.[7] imply, at least, that the structure of constraints in the asymmetric situation $W \neq V$ is more complicated: generators of $\mathcal{W}_K$-algebra (expressed in terms of t-variables) annihilate $Z_W\{t_k\}$ whenever $W[X]$ is a polynomial of power K. Our purpose below is to explore the origin of this important phenomenon in the most transparent way.

The natural derivation arises from comparison of eqs. (5) and (2). Indeed:

$$\frac{\partial Z_W}{\partial t_{p+1}} = \int D\Lambda\, e^{-\mathrm{tr}V[\Lambda]} \mathrm{tr}\Lambda^{p+1} \mathcal{F}\{\Lambda\} = \int D\Lambda\, e^{-\mathrm{tr}V[\Lambda]} \mathrm{tr}(\Lambda^p W'[\frac{\partial}{\partial \Lambda_{\mathrm{tr}}}]) \mathcal{F}\{\Lambda\}. \tag{7}$$

After integration by parts it turns into

$$\int D\Lambda \mathcal{F}\{\Lambda\}\mathrm{tr}(W'[-\frac{\partial}{\partial \Lambda_{tr}}]\Lambda^p)e^{-\mathrm{tr}V[\Lambda]} =$$

$$= \sum_{k>0}^{K} ks_k \int D\Lambda \mathcal{F}\{\Lambda\}\mathrm{tr}((-\frac{\partial}{\partial \Lambda_{\mathrm{tr}}})^{k-1}\Lambda^p)e^{-\mathrm{tr}V[\Lambda]}. \quad (8)$$

The leading term in the sum on the r.h.s., i.e.

$$Ks_K \int D\Lambda \mathcal{F}\{\Lambda\}\{\mathrm{tr}\,\Lambda^p(V'[\Lambda])^{K-1} + O(V^{K-2})\} =$$

$$= Ks_K \sum_{a_1,\ldots,a_{K-1}} a_1 t_{a_1\ldots a_{K-1}} t_{a_{K-1}} \frac{\partial}{\partial t_{a_1+\ldots+a_{K-1}+p+1-K}} Z_W\{t\} \quad (9)$$

i s a "classical" [1)] piece of the operator $\mathcal{W}^{(K)}_{p+1-K}$ — $(p+K-1)$-th harmonic of the spin-K generator of $\mathcal{W}_K$-algebra — acting on $Z_W\{t\}$. (Note also that according to (8) $p \geq 0$). This explains essentially the very origin of $\mathcal{W}_K$-constraint and its intimate relation to the form of potential $W[\Lambda\}$ in complete agreement with [7].

In general, the Ward identity (7) can be rewritten as a set of constraints

$$\left(-\frac{\partial}{\partial t_{p+1}} + \sum_{k>0}^{K} ks_k \tilde{\mathcal{W}}^{(k)}_{p+1-k}\{t\}\right) Z_W\{t\} = 0. \quad (10)$$

Operators $\tilde{\mathcal{W}}^{(k)}$ are defined by

$$\tilde{\mathcal{W}}^{(k)}_{p+1-k} e^{-\mathrm{tr}V[\Lambda]} = \mathrm{tr}((-\frac{\partial}{\partial \Lambda_{\mathrm{tr}}})^{k-1}\Lambda^p)e^{-\mathrm{tr}V[\Lambda]}. \quad (11)$$

They obey recurrent relation

$$\tilde{\mathcal{W}}^{(k+1)}_p = \sum_n nt_n \tilde{\mathcal{W}}^{(k)}_{n+p} + \sum_{a+b=p+k-1} \frac{\partial}{\partial t_a}\tilde{\mathcal{W}}^{(k)}_{b+1-k}; \quad p \geq -k \quad (12)$$

with

$$\tilde{\mathcal{W}}^{(2)}_p = \mathcal{L}_p = \sum_n nt_n \frac{\partial}{\partial t_{n+p}} + \sum_{a+b=p} \frac{\partial^2}{\partial t_a \partial t_b}; \quad p \geq -1, \quad (13)$$

$$\tilde{\mathcal{W}}^{(1)}_p = \mathcal{J}_p = \frac{\partial}{\partial t_p}; \quad p \geq 0. \quad (14)$$

Eqs.(12) and (13) imply that all the $\tilde{\mathcal{W}}^{(k)}$-operators are in fact proportional to linear combinations of Virasoro operators $\mathcal{L} = \tilde{\mathcal{W}}^{(2)}$. This may explain how in the continuum limit a single Ward identity (10) can give rise to entire set of constraints with lower spins. However, this topic is beyond the scope of this Letter. The continuum limit of these eqs. may be studied along the lines of [7].

3. While from the point of view of derivations it is illuminating first to study the simplest Ward identity (3) in the "1-matrix component" of 2-matrix model,

[1)]In the sense of ref.[9]

and then apply it to the (more sophisticated) analysis of 2-matrix case, it is of course possible to treat the resulting $\mathcal{W}$-constraints (10) as Ward identities in the entire 2-matrix model, related to the following infinitesimal change of integration variables

$$\delta X = \Lambda^p, \quad p \geq 0 \tag{15}$$

$$\delta\Lambda = \left(\sum_{m=0}^{K} ms_m \sum_{k=0}^{m-2} (-)^{k+1}(V')^{k+1}X^{m-2-k}\right)\Lambda^p + \text{"quantum corrections."}$$

This variation of variables induces the variation of potential:

$$\delta S = \left(\sum_{m=0}^{K} ms_m(-)^m(V')^{m-1}\right)\Lambda^p + \Lambda^{p+1} + \text{"quantum corrections"}. \tag{16}$$

The first term in this expression gives rise to the "classical" part of $\tilde{\mathcal{W}}$-algebra and the second one produces the derivative $\partial/\partial t_{p+1}$ in (10).

While the X-component of the variation (15) (which is of course nothing but (3)) does not change the integration measure $DXD\Lambda$, this is not true for Λ-component. The corresponding Jacobian is responsible for the "quantum" contributions to (15) and (16).

Eqs. (15)-(16) provide a possible generalization to the 2-matrix case of the derivation [9,10] of the Virasoro constraints in the discrete 1-matrix model from the Ward identities associated with the shift $M \to M + \epsilon M^{n+1}$ $(n \geq -1)$ of the integration variable.

4. Finally, we shall present some remarks.

First of all we should stress that eq.(11) defines only *positive* ($p \geq 1 - K$, to be exact) harmonics $\tilde{\mathcal{W}}_p^{(K)}$ of $\tilde{\mathcal{W}}^{(K)}$-operators.

Second, an important question is whether the set of $\tilde{\mathcal{W}}$-constraints (10) is closed. It is indeed the case. Namely it can be shown that

$$[\tilde{\mathcal{W}}_p^{(K)}, \tilde{\mathcal{W}}_q^{(K)}] \in Span\,\tilde{\mathcal{W}}_r^{(K)} \tag{17}$$

and $r \geq 1 - K$ as long as $p, q \geq 1 - K$. In particular,

$$[\tilde{\mathcal{W}}_p^{(2)}, \tilde{\mathcal{W}}_q^{(2)}] = (p-q)\tilde{\mathcal{W}}_{p+q}^{(2)}, \quad p, q \geq -1, \tag{18}$$

$$[\tilde{\mathcal{W}}_p^{(3)}, \tilde{\mathcal{W}}_q^{(3)}] = 2(p-q)\sum_k kt_k \tilde{\mathcal{W}}_{p+q+k}^{(3)} +$$

$$+\left[\sum_{a=0}^{p+1}(2p-q-2a) - \sum_{a=0}^{q+1}(2q-p-2a)\right]\frac{\partial}{\partial t_b}\tilde{\mathcal{W}}_{p+q-a}^{(3)}, \quad p, q \geq -2. \tag{19}$$

etc.

It deserves noting that (17) remains a highly non-trivial property of $\tilde{\mathcal{W}}$-algebra: the fact that this set is indeed closed makes an interesting exercise to observe the appearance of the entire tower of $\tilde{\mathcal{W}}^{(n)}$-constraints (with all spins

$n \leq K$) in the continuum limit from the single spin-K constraint at the discrete level.

Third, just the same $\tilde{\mathcal{W}}$-operators were found in [6] in a somewhat different context. It can be proved that they are really the same, thus demonstrating a kind of universal nature of $\tilde{\mathcal{W}}$-operators, at least, in the framework of discrete matrix models.

Fourth, the fact that the commutator of $\tilde{\mathcal{W}}^{(K)}$-operators in (17) is not just proportional to $\tilde{\mathcal{W}}_{p+q}^{(2K-2)}$ demonstrates that $\oplus_K \tilde{\mathcal{W}}^{(K)}$ *is not* a Lie algebra (to make it similar to $\mathcal{W}_\infty$, at least, the basis should be changed). This makes $\tilde{\mathcal{W}}$ even more similar to conventional $\mathcal{W}$-algebras which are also non-linear and closed as soon as only operators $\tilde{\mathcal{W}}^{(n)}$ of spins $n \leq K$ are considered.

5. To conclude, we demonstrated that the Ward-identities of Kontsevich-like models, derived in [6], are enough to obtain a closed (and presumably) complete set of Ward identities (loop equations) in discrete 2-Hermitean matrix model. These loop-equations involve $\tilde{\mathcal{W}}$-operators acting on one of potentials in the 2-matrix model, while the "highest" spin involved in these operators coincides with the power of another potential. So one can look at these constraints as at Ward identities of proper 1-matrix model (in terms of variable Λ) [2)] with suitable measure produced by integration over matrix X in (5). In principle, it is rather trivial statement that there exists algebra of $\tilde{\mathcal{W}}$-constraints in any 1-matrix model as it can be produced by conjugation from evident algebra (more precisely, Borel subalgebra) of operators cancelling the identity partition function [12] [3)]. This statement is the discrete counterpart of the suggestion of Gava and Narain [7], concerning the "asymmetric" (i.e. two potentials do not coincide) continuum limit of 2-matrix model, which is probably different from the symmetric limit, originally examined in [2].

At discrete level the $\tilde{\mathcal{W}}$-operators appear proportional to Virasoro operators, so that any solution to discrete Virasoro constraints automatically satisfies $\tilde{\mathcal{W}}_p^{(K)} Z = 0$ with $p \geq 1 - K$. The proper Ward identity in 2-matrix model, however, are linear combination of $\tilde{\mathcal{W}}_p^{(K)}$ and $\partial/\partial t_{p+K}$ with non-vanishing coefficients, so that actual constraints imposed on partition function of 2-matrix model are not expressible through Virasoro generators.

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[2)] Unlikely to "true" 2-matrix model $\mathcal{W}$-constraints depending on two sets of times, see the example of this in [11].

[3)] We certainly know the attempt to construct such constraints in more explicit way [13]. Unfortunately, it was done in the space of spectral parameter in terms of Baker-Akhiezer functions instead of more transparent language of partition (τ-) functions depending on time variables. We hope the present approach with the second potential truncated is more exposing.

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