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UNIFICATION OF ALL STRING MODELS WITH $c<1$ IN THE FRAMEWORK OF GENERALIZED KONTSEVICH MODEL

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A 1-matrix model is proposed, which nicely interpolates between double-scaling continuum limits of all multimatrix models. The interpolating partition function is always a KP τ-function and always obeys $\mathcal{L}_{-1}$-constraint and string equation. Therefore this model can be considered as a natural unification of all models of $2d$-gravity (string models) with $c \leq 1$.

The model. The purpose of this letter is to introduce a new theory, which we call *Generalized Kontsevich's Model* (GKM) and to describe its structure and appealing properties. The partition function of the GKM is defined by the following integral over $N \times N$ Hermitean matrix:

$$Z_N^{\{\mathcal{V}\}}[M] \equiv \frac{\int e^{U(M,Y)}dY}{\int e^{-U_2(M,Y)}dY}, \tag{1}$$

where

$$U(M,Y) = Tr[\mathcal{V}(M+Y) - \mathcal{V}(M) - \mathcal{V}'(M)Y] \tag{2}$$

and

$$U_2(M,Y) = \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} U(M,\epsilon Y), \tag{3}$$

is an Y^2-term in U. M is also a Hermitean $N \times N$ matrix with eigenvalues $\{\mu_i\}$, $\mathcal{V}(\mu)$ is arbitrary analytic function.

Integrable structure. After the shift of variables $X = Y + M$ and integration over angular components of X, $Z_N^{\{\mathcal{V}\}}[M]$ acquires the form of

$$Z_N^{\{\mathcal{V}\}}[M] = \frac{[\det \tilde{\Phi}_i(\mu_j)]}{\Delta(M)}, \tag{4}$$

where $\Delta(M) = \prod_{i<j}(\mu_i - \mu_j)$ is the Van-der-Monde determinant, and functions

$$\tilde{\Phi}_i(\mu) = [\mathcal{V}''(\mu)]^{1/2} e^{\mathcal{V}(\mu) - \mu\mathcal{V}'(\mu)} \int e^{-\mathcal{V}(x) + x\mathcal{V}'(\mu)} x^i dx \tag{5}$$

The only assumption necessary for the derivation of (4) from (1) is the possibility to represent the potential $\mathcal{V}(\mu)$ as a formal series in positive *integer* powers of μ.

Formula (4) with *arbitrary* entries $\phi_i(\mu)$ is characteristic for generic KP τ-function $\tau^G(T_n)$ in Miwa's coordinates

$$T_n = \frac{1}{n}\mathrm{Tr}M^{-n}, \quad n \geq 1 \tag{6}$$

and the point G of Grassmannian is defined by potential $\mathcal{V}$ through the set of basis vectors $\{\phi_i(\mu)\}$. (Let us remind that *a priori* definition is $\tau^G(T_n) = <0|e^{\sum T_n J_n} G|0>$, where J stands for the free-fermion $U(1)$ current and G is an exponent of quadratic combination of free fermion operators.) Therefore

$$Z^{\{\mathcal{V}\}}[M] = \tau^{\{\mathcal{V}\}}(T_n). \tag{7}$$

The case of finite N in this formalism is distinguished by the condition that only N of the parameters $\{\mu_i\}$ are finite. In order to take the limit $N \to \infty$ in the GKM (1) it is enough to bring all the $\mu_i^! s$ from infinity. In this sense this a smooth limit, in contrast to the singular conventional double-scaling limit, which one needs to take in ordinary (multi)matrix models.

$\mathcal{L}_{-1}$-constraint. The set of function $\{\tilde{\Phi}_i(\mu)\}$ in (4) is, however, not arbitrary. They are all expressed through a single function – potential $\mathcal{V}(\mu)$, –*and* are in fact recurrently related: if we denote the integral in (5) through $F_i(\mathcal{V}'(\mu))$, then

$$F_i(\lambda) = (\partial/\partial\lambda)^{i-1} F_1(\lambda). \tag{8}$$

This relation is enough to prove, that

$$\frac{\partial}{\partial T_1}\log\ Z_N^{\{\mathcal{V}\}} = \mathrm{Tr}M - \mathrm{Tr}\frac{\partial}{\partial \Lambda_{tr}}\log\det F_i(\lambda_j) \tag{9}$$

whenever potential $\mathcal{V}(\mu)$ grows faster than μ as $\mu \to \infty$.

Thus, $Z^{\{\mathcal{V}\}}$ satisfies a simple identity:

$$\frac{1}{Z^{\{\mathcal{V}\}}}\mathcal{L}_{-1}^{\{\mathcal{V}\}} Z_N^{\{\mathcal{V}\}} = \frac{\partial}{\partial T_1}\log Z_N^{\{\mathcal{V}\}} - \mathrm{Tr}M + \mathrm{Tr}\frac{\partial}{\partial \Lambda_{tr}}\log\det F_i(\lambda_j) = 0 \tag{10}$$

where operator $\mathcal{L}_{-1}^{\{\mathcal{V}\}}$ is defined to be

$$\mathcal{L}_{-1}^{\{\mathcal{V}\}} = \sum_{n\geq 1}\mathrm{Tr}[\frac{1}{\mathcal{V}''(M)M^{n+1}}]\frac{\partial}{\partial T_n} +$$

$$+\frac{1}{2}\sum_{i,j}\frac{1}{\mathcal{V}''(\mu_i)\mathcal{V}''(\mu_j)}\frac{\mathcal{V}''(\mu_i) - \mathcal{V}''(\mu_j)}{\mu_i - \mu_j} - \frac{\partial}{\partial T_1} \tag{11}$$

(the items with $i = j$ are included into the sum). The reason why this operator is denoted by $\mathcal{L}_{-1}$ will be clear after reductions of GKM will be discussed. From eqs.(9),(10) it follows, that partition function of GKM usually satisfies the constraint

$$\mathcal{L}_{-1}^{\{\mathcal{V}\}}\tau^{\{\mathcal{V}\}} = 0. \tag{12}$$

Reductions. The integral $\mathcal{F}^{\{\mathcal{V}\}}[\Lambda]$, $\Lambda \equiv \mathcal{V}'(M)$, in the numerator of (1) satisfies the Ward identity

$$\mathrm{Tr}\left\{\epsilon(\Lambda)\left[\mathcal{V}'\left(\frac{\partial}{\partial \Lambda_{\mathrm{tr}}}\right) - \Lambda\right]\right\}\mathcal{F}_N^{\{\mathcal{V}\}} = 0 \tag{13}$$

(as result of invariance under any shift of integration variable $X \to X + \epsilon(M)$). If potential $\mathcal{V}(\mu)$ is restricted to be a polynomial of degree $K+1$, this identity implies, that the functions (8) obey additional relations:

$$F_{m+Kn}(\lambda) = \lambda^n \cdot F_m(\lambda) + \sum_{i=1}^{m+Kn-1} s_i F_i(\lambda). \tag{14}$$

Since the sum at the *r.h.s.* does not contribute to determinant (5), we can say that all the functions F_n are expressed through the first K functions $F_1 ... F_K$ by multiplication by powers of $\lambda = \mathcal{V}'(\mu)$. Such situation (when the basis vectors ϕ_i, defining the point of Grassmannian are proportional to the first K ones) corresponds to reduction of KP-hierarchy. This reduction depends on the form of $\mathcal{V}'(\mu)$ and in the case of $\mathcal{V}(\mu) = \mathcal{V}_K(\mu) = const \cdot \mu^{K+1}$ coincides with the well-known K-reduction of the KP-hierarchy (KdV as $K=2$, Boussinesq as $K=3$ etc.). Thus in such cases partition function of GKM becomes $\tau^{\{K\}}$-function of the corresponding hierarchy. Moreover, in this case one can represent each function $\tilde{\Phi}_i(\mu)$ in (5) in the form of $A^{i-1}\tilde{\Phi}_1(\mu)$, where A is proper differential operator which can be read out from the formulas (5) and (8). Suprisingly, this operator coincides with that from the paper [1] implementing different approach based on transmuting Virasoro ($\mathcal{W}$-) constraints into the conditions on the point of Grassmannian.

Generic $\tau^{\{K\}}$ possesses an important property: it is almost independent of all time-variables T_{nK}. To be exact,

$$\partial \log \tau^{\{K\}} / \partial T_{nK} = a_n = \mathrm{const} \tag{15}$$

If $\mathcal{V} = \mathcal{V}_K$, the generic expression (12) for the $\mathcal{L}_{-1}$-operator turns into

$$\mathcal{L}_{-1}^{\{K\}} = \frac{1}{K}\sum_{n>K} nT_n \partial/\partial T_{n-K} + \frac{1}{2K}\sum_{\substack{a+b=K\\ a,b>0}} aT_a bT_b - \partial/\partial T_1 \tag{16}$$

The last item at the *r.h.s.* may be eliminated by the shift of time-variables:

$$T_n \to \hat{T}_n^{\{K\}} = T_n - \frac{K}{n}\delta_{n,K+1}. \tag{17}$$

This shift is, however, K-dependent and does not seem to have too much sense. However, only expressed in terms of these $\hat{T}$'s the constraint (12) acquires the form of

$$\mathcal{L}_{-1}^{\{K\}}\tau^{\{K\}} = \left\{\frac{1}{K}\sum_{\substack{n>K\\ n\neq 0 \bmod K}} n\hat{T}_n \partial/\partial \hat{T}_{n-K} + \frac{1}{2K}\sum_{\substack{a+b=K\\ a,b>0}} a\hat{T}_a b\hat{T}_b\right\}\tau^{\{K\}} =$$

$$= \sum_n a_n(n+1)\hat{T}_{(n+1)K}\tau^{\{K\}}. \tag{18}$$

with the *l.h.s.* familiar from [2]. The sum at the *r.h.s.* of (18) does not contribute to the "string equation"

$$\frac{\partial}{\partial T_1}\frac{\mathcal{L}_{-1}^{\{K\}}\tau^{\{K\}}}{\tau^{\{K\}}} = 0. \tag{19}$$

Moreover, in variance with generic $\tau^{\{K\}}$ the partition function $Z^{\{K\}}$ of GKM is expected to obey (15) and (18) with all $a_n = 0$.

Universal string equation. Generalization of (19) to the case of arbitrary potential

$$\frac{\partial}{\partial T_1}\frac{\mathcal{L}_{-1}^{\{\mathcal{V}\}}\tau^{\{\mathcal{V}\}}}{\tau^{\{\mathcal{V}\}}} = 0. \tag{20}$$

may be transformed to the following form

$$\sum_{n\geq -1}\mathcal{T}_n\frac{\partial^2\log\tau}{\partial T_1\partial T_n} = u, \tag{21}$$

where

$$\mathcal{T}_n \equiv Tr\frac{1}{\mathcal{V}''(\mathcal{M})}\frac{1}{\mathcal{M}^{n+1}}, \tag{22}$$

$$u \equiv \frac{\partial^2\log\tau}{(\partial T_1)^2}, \quad \frac{\partial\log\tau}{\partial T_0} \equiv 0, \quad \frac{\partial\log\tau}{\partial T_{-1}} \equiv T_1.$$

If Baker-Akhiezer are introduced:

$$\Psi_\pm(z|T_k) = e^{\sum T_k z^k}\frac{\tau(T_n \pm \frac{z^{-n}}{n})}{\tau(T_n)}, \tag{23}$$

string equation (22) can be rewritten in the form of bilinear relation

$$\sum_i \frac{\Psi_+(\mu_i)\Psi_-(\mu_i)}{\mathcal{V}''(\mu_i)} = u. \tag{24}$$

$\mathcal{W}$-constraints. According to the arguments of refs. [2] the constraint

$$\mathcal{L}_{-1}^{\{K\}}\tau^{\{K\}} = 0 \tag{25}$$

(*i.e.* (18) with the vanishing *r.h.s.*, as it is in fact the case if we deal with the model (1)) implies the entire tower of $\mathcal{W}$-constraints

$$\mathcal{W}_{Kn}^{(k)}Z^{\{K\}} = 0, \quad k = 2,3,...,K; \quad n \geq 1-k \tag{26}$$

imposed on $\tau^{\{K\}}$. Here $\mathcal{W}_{Kn}^{(p)}$ is the $n-th$ harmonics of the $p-th$ generator of Zamolodchikov's W_K-algebra (the proper notation would be $\mathcal{W}_n^{(p)\{K\}}$, but it is a bit too complicated). There is a Virasoro Lie sub-algebra, generated by $\mathcal{W}_{Kn}^{(2)} = \mathcal{L}_n^{\{K\}}$, and the particular $\mathcal{L}_{-1}^{\{K\}}$ is just the operator (16). This is the origin of our notation $\mathcal{L}_{-1}^{\{\mathcal{V}\}}$ in the generic situation (where the entire Virasoro subalgebra of W_∞ was not explicitly specified).

Besides being a corollary of (24), the constraints (25) can be directly deduced from the Ward identity (13). For the case of $K = 2$ (which is original Kontsevich's model [3]) this derivation was given in ref. [4] (see also [5,6] for alternative proofs). Unfortunately, for $K \geq 3$ the direct corollary of (13) is not just (25), but peculiar linear combinations of these constraints, *e.g.* for $K = 3$ they look like

$$\mathcal{W}_{3n}^{(3)} Z_\infty^{\{3\}} = 0, \ n \geq -2;$$

$$\left\{ \sum_{k\geq 1} (3k-1)\hat{T}_{3k-1} \mathcal{W}_{3k+3n}^{(2)} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+2}} \mathcal{W}_{3b-3}^{(2)} \right\} Z_\infty^{\{3\}} = 0, \ a,b \geq 0, \ n \geq -2;$$

$$\left\{ \sum_{k\geq 1+\delta_{n+3,0}} (3k-2)\hat{T}_{3k-2} \mathcal{W}_{3k+3n}^{(2)} + \sum_{a+b=3n} \frac{\partial}{\partial T_{3a+1}} \mathcal{W}_{3b-3}^{(2)} \right\} Z_\infty^{\{3\}} = 0, \ a,b \geq 0, \ n \geq -3. \tag{27}$$

For identification of (26) with (25) one can argue, that both sets of constraints possess unique, and thus coinciding, solutions.

Multimatrix models. While detailed investigation of the properties of multimatrix models in the double-scaling limit (the analogue of ref. [7] in the case of conventional Hermitean 1-matrix model) is still lacking, it has been suggested in [2] that the square roots of their partition functions, $\sqrt{\Gamma_{ds}^{\{K-1\}}}$ ($K-1$ is the number of matrices, index ds means, that partition function is considered in the double scaling limit), possess the following properties:

$$\mathcal{W}_{Kn}^{(k)} \sqrt{\Gamma_{ds}^{\{K-1\}}} = 0 \quad , \quad k = 2,3,...,K; \quad n \geq 1-k. \tag{28}$$

Comparing these properties to the above information about GKM, we obtain:

$$Z_\infty^{\{K\}} = \sqrt{\Gamma_{ds}^{\{K-1\}}} \tag{29}$$

Conclusion. To conclude, we presented a brief description of the properties of the GKM, defined by eq.(1). Its partition function may be considered as a functional of two different variables: potential $\mathcal{V}(\mu)$ and the infinite-dimensional Hermitean matrix M with eigenvalues $\{\mu_i\}$. Partition function $Z_N^{\{\mathcal{V}\}}$ is an N-independent KP τ-function, considered as a function of time-variables $T_n = \frac{1}{n} Tr M^{-n}$ and the point of Grassmannian is specified by the choice of potential. The N-dependence enters only through the argument M : we return to finite-dimensional matrices if only N eigenvalues of M are finite. In this sense the "continuum" limit of $N \to \infty$ is smooth.

The GKM is associated with a subset of Grassmannian, specified by additional $\mathcal{L}_{-1}$-constraint (12). For particularly adjusted potentials $\mathcal{V}(\mu) = const \cdot \mu^{K+1}$, the corresponding points in Grassmannian lies in the subvarieties, associated with K-reductions of KP-hierarchy, $Z^{\{\mathcal{V}\}}$ becomes independent of all the time-variables T_{Kn}, and the $\mathcal{L}_{-1}$-constraint implies the whole tower of W_K-algebra constraints on the reduced τ-function. These properties are exactly the same as suggested for double scaling limit of the $K-1$-matrix model, and in fact there is an identification (29).

All this means, that GKM provides an interpolation between double-scaling continuum limits of all multimatrix models and thus between all string models with $c \leq 1$. Moreover, this is a reasonable interpolation, because both integrable and "string-equation" structures are preserved. This is why we advertise GKM as a plausible (on-shell) prototype of a unified theory of 2d gravity. All the proofs will be presented in ref. [8].

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