Adiabatic growing, multistability, and control of soliton-comb states in $\chi^{(2)}$ microresonators for pumping into second-harmonic modes[1)]

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Transfer of the soliton-comb concept established for $\chi^{(3)}$ microresonators [1, 2] to $\chi^{(2)}$ resonators represents an important problem [3, 4]. Within this concept, a dual FH-SH soliton propagates in a pumped resonator with velocity v_0 due to cascaded SH generation and OPO processes. The pump wavelength λ_p satisfies phase-matching conditions, and the soliton is dissipative – it balances not only dispersion broadening and nonlinear compression, but also gain and losses. Minimization of the FH-SH group velocities difference is crucial [4]; the radial poling of the resonator can be used to get it [5].

Nonlinear equations governing the dependence of FH and SH envelopes (F and S) on the azimuth angle φ and time t are known [3, 4]. Two schemes with pumping into a FH (or SH) resonator mode are envisaged. For the FH pumping, both F and S are 2π-periodic in φ. In the SH pumping case, also topologically different antiperiodic solutions with $F(\varphi,t) = -F(\varphi+2\pi,t)$ and $S(\varphi,t) = S(\varphi+2\pi,t)$ are allowed [4]. To realize the periodic (P) and antiperiodic (A) states, it is necessary to pump SH modes with even and odd azimuth numbers.

Single-soliton P-states in $\chi^{(2)}$ resonators were demonstrated numerically at the zero walk-off point [3]. The found solitons are locally stable, but not easily accessible. More recently, the existence of single-soliton A-states was established [4]. These states were found to be accessible upon switching the pump on. Neither analytical examples of the $\chi^{(2)}$ solitons, nor general numerical methods for their search are known so far. Experimentally, the necessary zero walk-off point vicinity is not realized – solitons detected in $\chi^{(2)}$ resonators [6] are due to Kerr nonlinearity.

We offer a regular method for the search of stable and accessible $\chi^{(2)}$ soliton-comb solutions. It allows us to get new single- and multi-soliton states and to demonstrate multistability of different regimes. It is based on a strong link between the spatial properties of near-threshold and well-above-threshold steady-state solutions for F and S. The first ones include typically not many significant FH and SH modes. Applying numerical procedure of adiabatically slow rise of the pump power and increasing the number of modes, it is possible to get numerous developed soliton-comb states. These states are new, stable and accessible.

Specifically, we employ instead of F, S their normalized Fourier harmonics f_j, s_l. In the SH pumping case, they obey evolutional nonlinear equations

$$\dot{f}_j + (1 + i\delta_1 + i\beta_1 j^2) f_j = -i \sum_l s_l f^*_{l-j} \qquad (1)$$

$$r\dot{s}_l + (1 + i\delta_2 + i\beta_2 l^2 - i\alpha l) s_l = \eta\delta_{l,0} - i \sum_j f_j f_{l-j}.$$

Here $\beta_{1,2} = d_{1,2}/\gamma_{1,2}R^2$ and $\alpha = (v_1 - v_2)/\gamma_2 R$ are the normalized dispersion and walk-off coefficients, R is the major resonator radius, $\gamma_{1,2}$ are the decay rates for the FH, SH modes, $v_{1,2}$ and $d_{1,2}$ are the known group velocities and their dispersions, $r = \gamma_1/\gamma_2$, $\delta_{1,2}$ are the normalized FH and SH frequency detunings, η is the normalized pump amplitude, and the dot indicates differentiation in the normalized time $\tau = \gamma_1 t$. Set (1) is written for a coordinate frame moving with velocity v_1. For mm-sized resonators, typically $|\beta_{1,2}| \ll 1$ and $|\alpha| \gg 1$. The case $|\alpha| \lesssim 1$ is relevant to a close vicinity of zero walk-off point λ_c. The main variable parameters are η, $\delta_{1,2}$, and λ_p. Set (1) is valid for both P- and A-cases. In the P-case we have $l, j = 0, \pm 1, \pm 2, \ldots$. In the A-case l takes the same values, while $j = \pm 1/2, \pm 3/2, \ldots$.

Truncated set (1) was solved numerically with different initial conditions (ICs) for the amplitudes. The total number of FH + SH modes ranged from 16 to 1024. Correctness of the results was verified via increasing mode

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number and decreasing time step. Establishment of the steady state was controlled with a high accuracy [4]. To quantify $\beta_{1,2}$ and α in the vicinity of λ_c spectral point, we set exemplarily $2\pi R = 1\,\mathrm{cm}$, $\gamma_{1,2} = 3 \times 10^7\,\mathrm{s}^{-1}$, and used dependences $v_{1,2}(\lambda_p)$, $d_{1,2}(\lambda_p)$ relevant to $LiNbO_3$ crystals. This gives $\beta_1 \simeq 5.3 \times 10^{-3}$, $\beta_2 \simeq -1.44 \times 10^{-3}$, and $\alpha \simeq -0.13 \times \delta\lambda\,[\mathrm{nm}]$.

Consider some results for the P-case at $\alpha = 0$, $\delta_2 = 0$, $\delta_1 = \delta_1^{(j)} = -\beta_1 j^2$ with $j = 1$. Using ICs with random phases, such that $|f_j(0)| \ll 1$ and $|s_l(0)| \ll 1$, we arrive after a transient stage at a steady state with $|s_0| \simeq 1$ and very small other harmonics. After that, we employ our adiabatic procedure: The pump amplitude η is slightly increased and the steady-state amplitudes f_j, s_l are used as new ICs. After establishment of a new steady state, the procedure was repeated many times. The number of harmonics was gradually increased as well. In the whole investigated range $1 < \eta \le 30$ we obtained a continuous family of nonlinear states $f(\eta, \varphi)$, $s(\eta, \varphi)$ that have to be qualified as dual two-soliton states. The relative soliton velocity $v_{01} = v_0 - v_1$ at $\alpha = 0$ is zero, and the absolute velocity is $v_1 \simeq 1.37 \times 10^{10}\,\mathrm{cm/s}$.

Figure 1 shows the periodic 2-soliton state for $\eta = 20$. We see from (a) and (d) that each of the intensity profiles $|f|^2(\varphi)$ and $|s|^2(\varphi)$ consists of two narrow

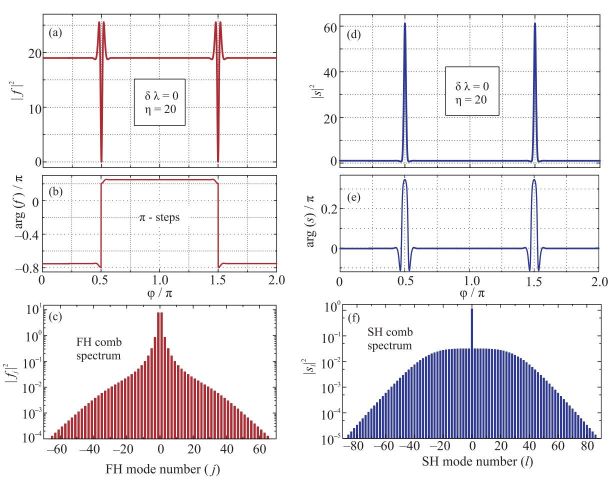

Fig. 1. (Color online) Periodic two-soliton state at $\alpha = 0$ and $\eta = 20$ grown adiabatically for $\delta_2 = 0$ and $\delta_1 = -\beta_1$. (a), (d) – The FH, SH intensity profiles; (b), (e) – the corresponding phase profiles; (c), (f) – the FH, SH comb spectra

π-spaced symmetric peaks. The peaks possess the backgrounds $|\bar{f}|^2$ and $|\bar{s}|^2$. Behavior of the phases $\arg(f)$ and $\arg(s)$ is remarkable. While the SH phase experiences modest deviations when crossing the soliton area, the FH phase shows π-steps. Thus, $f(\varphi)$ changes sign when crossing the soliton area and our periodic 2-soliton state consists of two A-solitons. This structure emerges spontaneously without imposing the antiperiodic conditions. The FH and SH comb spectra are shown in (c) and (f). These spectra are well developed and symmetric, and $|s_0| \simeq 1$. Owing to the π-periodicity in φ, nonzero FH and SH harmonics are $j = \pm 1, \pm 3, \ldots$ and $l = 0, \pm 2, \ldots$. The comb line spacing is here $2v_1/R$.

When switching $|j|$ to 2 and 3, we arrive at 4- and 6-soliton P-states consisting of A-soliton pairs. Also, we varied slowly detuning δ_1 within the broad range $[-1, 1]$ starting from well developed two-soliton states for $\eta \gtrsim 10$. The dual symmetric soliton solution survives during this adiabatic procedure; the soliton amplitudes and the widths of the comb spectra decrease only modestly when changing $|\delta_1|$. Thus, different stable P-states can exist at the same external parameters η and $\delta_{1,2}$. Also, we applied our adiabatic procedure to the case of nonzero walk-off, $\alpha \neq 0$. The intensity profiles have here pronounced asymmetric oscillating tails and become less localized and intense as compared to the profiles of Fig. 1, while $v_{01} \neq 0$. Similarly, we obtain 1-, 3-, and 5-soliton A-states.

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