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Gauge equivalence between 1 + 1 rational Calogero–Moser field theory and higher rank Landau–Lifshitz equation

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The $1+1$ field generalization of the Calogero–Moser model was proposed in [1, 2], see also [3]. The Hamiltonian is given by the following expression:

$$
\begin{gathered}
\mathcal{H}^{\rm 2dCM}=\oint \mathrm{d}x\, H^{\rm 2dCM}(x),\\
H^{\rm 2dCM}(x)=\sum_{i=1}^{N} p_i^2\,(c-kq_{ix})-\\
-\frac{1}{Nc}\left(\sum_{i=1}^{N} p_i\,(c-kq_{ix})\right)^2-\\
-\sum_{i=1}^{N}\frac{k^4 q_{ixx}^2}{4\,(c-kq_{ix})}+\frac{k^3}{2}\sum_{i\neq j}^{N}\frac{q_{ix}q_{jxx}-q_{jx}q_{ixx}}{q_i-q_j}-\\
-\frac{1}{2}\sum_{i\neq j}^{N}\frac{1}{(q_i-q_j)^2}\Big[\,(c-kq_{ix})^2\,(c-kq_{jx})+\\
+(c-kq_{ix})\,(c-kq_{jx})^2-ck^2\,(q_{ix}-q_{jx})^2\,\Big],
\end{gathered}
\tag{1}
$$

where x is the (space) field variable and $k\in\mathbb{C}$ is a constant parameter. The momenta p_i and coordinates q_j are canonically conjugated fields: $\{q_i(x),p_j(y)\}=\delta_{ij}\delta(x-y)$. The model (1) is integrable in the sense that it has algebro-geometric solutions and equations of motion are represented in the Zakharov–Shabat (or Lax or zero curvature) form: $\partial_t U(z)-k\partial_x V(z)+[U(z),V(z)]=0$, where U-V pair $U^{\rm 2dCM}(z),V^{\rm 2dCM}(z)\in \mathrm{Mat}(N,\mathbb{C})$ is a pair of matrix valued functions of the fields $p_j(x)$, $q_j(x)$, $j=1,...,N$ and their derivatives. They also depend on the spectral parameter z. Explicit expression for U-V pair can be found in [1, 2]. It was argued in [3] that there exist a gauge transformation $G(z)\in \mathrm{Mat}(N,\mathbb{C})$, which transforms U-V pair for the field Calogero–Moser model to the one for some Landau–Lifshitz type model:

$$
U^{\rm LL}(z)=G(z)U^{\rm 2dCM}(z)G^{-1}(z)+k\partial_x G(z)G^{-1}(z). \tag{2}
$$

For the $N=2$ case explicit construction of the matrix $G(z)$ and the change of variables was derived in our paper [4], and the Landau–Lifshitz model for GL_2 rational R-matrix was derived in [5]. The goal of this article is to define the gauge transformation in gl_N case, describe the corresponding Landau–Lifshitz type model and find explicit change of variables using relation (2).

Recently the $1+1$ field generalization of the so-called rational top model was suggested in [6]. It is given by Landau–Lifshitz type equation, i.e. the field variables are arranged into $N\times N$ matrix S and the Poisson structure is linear: $\{S_{ij}(x),S_{kl}(y)\}=N^{-1}\Big(S_{il}(x)\delta_{kj}-S_{kj}(x)\delta_{il}\Big)\delta(x-y)$. The construction of the Landau–Lifshitz equation and its U-V pair is based on R-matrix satisfying the associative Yang–Baxter equation [7, 8]: $R^\hbar_{12}R^\eta_{23}=R^\eta_{13}R^{\hbar-\eta}_{12}+R^{\eta-\hbar}_{23}R^\hbar_{13}$, $R^x_{ab}=R^x_{ab}(z_a-z_b)$. Suppose $\mathrm{rank}(S)=1$, so that $S^2=cS$, $c=\mathrm{tr}(S)$. Then the Landau–Lifshitz equation reads:

$$
\partial_t S=k^{-2}c\,[S,\partial_x^2 S]+2c\,[S,J(S)]-2k[S,E(\partial_x S)]\,, \tag{3}
$$

where $E(S)=\mathrm{tr}_2\Big(r^{(0)}_{12}\overset{2}{S}\Big)$, $\overset{2}{S}=1_N\otimes S$ and $J(S)=\mathrm{tr}_2\Big(m_{12}(0)\overset{2}{S}\Big)$ are defined through the coefficients of R-matrix expansion in the classical limit $R^\hbar_{12}(z)=\hbar^{-1}1_N\otimes 1_N+r_{12}(z)+\hbar\, m_{12}(z)+O(\hbar^2)$ and $r^{(0)}_{12}$ is the coefficient in the expansion $r_{12}(z)=z^{-1}P_{12}+r^{(0)}_{12}+O(z)$, where P_{12} is the permutation operator. Equa-

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tions (3) are Hamiltonian with the following Hamiltonian function:

$$H^{\rm LL} = \oint dy \Big(cN \mathrm{tr}\Big(S\, J(S)\Big) - \frac{Nk^2}{2c}\, \mathrm{tr}\Big(\partial_y S\, \partial_y S\Big) + \\ + kN \mathrm{tr}\Big(\partial_y S\, E(S)\Big)\Big), \quad S = S(y). \qquad (4)$$

In this paper we use the rational R-matrix calculated in [9]. In the $N = 2$ case it reproduces the 11-vertex R-matrix found by I. Cherednik [10]. For $N > 2$ all its properties, different possible forms and explicit expressions for the coefficients of expansions near $z = 0$ and $\hbar = 0$ can be found in [11].

The statement is that by applying the gauge transformation with a certain matrix $G(z)$ we obtain the desired relation (2). Calculations are performed similarly to those in $0+1$ mechanics [12]. As a result we obtain explicit change of variables expressed through elementary symmetric functions σ_k:

$$S_{ij} = \frac{(-1)^{\varrho(j)+1}}{N} \times \\ \times \sum_{m=1}^{N} \frac{(q_m)^{\varrho(i)}(\tilde{p}_m + \frac{k\alpha_{mx}}{\alpha_m}) + \alpha_m^2 \varrho(i)(q_m)^{\varrho(i)-1}}{\prod_{l\neq m}(q_m - q_l)}\, \sigma_{\varrho(j)}(q),$$

$$\tilde{p}_j = p_j - \sum_{l\neq j}^{N} \frac{\alpha_j^2}{q_j - q_l} \qquad (5)$$

(here $\varrho(i) = i-1$ for $i \le N-1$ and $\varrho(i) = N$ for $i = N$) with the properties

$$\mathrm{Spec}(S) = (0, ..., 0, c), \ \ \mathrm{rk}(S) = 1, \ \ \mathrm{tr}(S) = c, \ \ S^2 = cS, \qquad (6)$$

where $\alpha_i^2 = kq_{ix} - c$. It can be also verified that the Poisson brackets for $S_{ij}(p, q, c)$ calculated through the canonical brackets for p_i, q_j indeed reproduce the linear Poisson structure, so that (5) is a Poisson map. The Hamiltonian (1) of $1+1$ field Calogero–Moser model coincides with the one (4) for the Landau–Lifshitz equation under the change of variables (5): $H^{\rm LL}[S(p(x), q(x))] = H^{\rm 2dCM}[p(x), q(x)]$.

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