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Exact relations between running of α_s and α in $\mathcal{N} = 1$ SQCD + SQED

A. L. Kataev[+*1)], *K. V. Stepanyantz*[×]

[+]*Institute for Nuclear Research of the Russian Academy of Sciences, 117312 Moscow, Russia*

[*]*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Russia*

[×]*Moscow State University, Physical Faculty, Department of Theoretical Physics, 119991 Moscow, Russia*

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In $\mathcal{N} = 1$ supersymmetric QCD and QED theory we derive the (all-order) exact equations relating the renormalization group behaviour of the strong and electromagnetic couplings and prove that they are valid in the combined higher covariant derivative regularized and minimal subtraction of logarithms renormalized prescription. In particular, the β-function of $\mathcal{N} = 1$ supersymmetric QCD can be expressed in terms of the Adler D-function. If all flavors have the same absolute value of the electromagnetic charges, it is also possible to write a simple relation between the β-functions for the strong and electromagnetic coupling constants. In this particular case there is a special renormalization group invariant relation.

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The renormalization group method [1–3] plays an important role in physical considerations. It enables one to study the problems of scale and scheme dependence of the perturbation theory predictions for Green functions and for the Adler D-function [4] in particular. These studies allows us to investigate the consequences of taking into account the contributions of different strong interactions to different inclusive processes generated by one-photon e^+e^--annihilation.

In $\mathcal{N} = 1$ supersymmetric gauge theories, there are statements that can be considered as nonrenormalization theorems. Their applications provide a theoretical justification for supports of the property of the relation of the corresponding gauge β-function to the anomalous dimension of matter superfields through the Novikov, Shifman, Vainshtein and Zakharov (NSVZ) equation [5, 6]. Although this equation is not valid in the $\overline{\mathrm{DR}}$ scheme [7] formulated in D-dimensions, it turns out that the NSVZ schemes, in which the corresponding strict relations are restored at the renormalized level after the finite renormalizations considered in [6, 8], can be constructed using the Slavnov higher covariant derivative (HD) regularization [9, 10], typical for four dimensions, and the procedure of minimal subtraction of logarithms (MSL) [11]. In the resulting HD + MSL renormalization scheme, formulated there for the theory regularized by higher covariant derivatives, one is dealing with the powers of $\ln \Lambda/\mu$ (where Λ is the ultraviolet cut-off and μ is the renormalization point), which are included into the renormalization constants. The whole class of NSVZ schemes was described in [12, 13] and forms a *subgroup* of the renormalization group [13].

The NSVZ-like equation was also constructed and proved in all orders for the Adler D-function in $\mathcal{N} = 1$ supersymmetric chromodynamics (SQCD) interacting with the supersymmetric electromagnetic field [14, 15]. It looks similar to the similar equation for $\mathcal{N} = 1$ supersymmetric electrodynamics (SQED) [16, 17] and is closely related to the NSVZ β-functions of the non-Abelian Yang–Mills theories with multiple gauge couplings considered in [18, 19].

In this paper, we note that the resemblance between the NSVZ equations for the β-functions and the Adler D-function in $\mathcal{N} = 1$ SQCD implies that the renormalization group running of strong and electromagnetic couplings of the combined SQED + SQCD model are related by certain exact equations.

Let us consider a massless supersymmetric QCD-like theory, that is invariant under gauge group $G \times U(1)$ transformations where G is a simple group generalizing $SU(3)$ for usual QCD and $U(1)$ describes the supersymmetric analog of the electromagnetic interaction. The corresponding renormalization group functions are defined as

1)e-mail: kataev@ms2.inr.ac.ru

$$\beta_s(\alpha_s,\alpha) \equiv \frac{d\alpha_s}{d\ln\mu}\Big|_{\alpha_0,\alpha_{s0}=\mathrm{const}};$$
$$\beta(\alpha,\alpha_s) \equiv \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_0,\alpha_{s0}=\mathrm{const}}; \qquad (1)$$
$$\gamma(\alpha_s,\alpha) \equiv \frac{d\ln(Z)}{d\ln\mu}\Big|_{\alpha_0,\alpha_{s0}=\mathrm{const}},$$

where Z is the renormalization constant of matter superfields. The equations for the β-functions of both coupling constants in the model under consideration were constructed in [18] and read:

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}\Big|_{\mathrm{NSVZ}} = -\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\times \times\Big[3C_2 - 2T(R)N_f\Big(1-\gamma(\alpha_s,\alpha)\Big)\Big]; \qquad (2)$$
$$\frac{\beta(\alpha,\alpha_s)}{\alpha^2}\Big|_{\mathrm{NSVZ}} = \frac{1}{\pi}\dim R\, N_f\Big(1-\gamma(\alpha_s,\alpha)\Big), \qquad (3)$$

where $\dim R$ is the dimension of the representation of the simple group G, $C(R)$ is its Casimir operator and $T(R)$ is the Dynkin index, while N_f is the number of "coloured" species of the corresponding matter fields. Comparing Eqs. (2) and (3), one can be eliminate from them the anomalous dimension and thus relate the β-functions for the non-Abelian and Abelian supersymmetric coupling constants by the following all-order exact equation:

$$\Big(1-\frac{C_2\alpha_s}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}\Big|_{\mathrm{NSVZ}} = = -\frac{3C_2}{2\pi} + \frac{T(R)}{\dim R}\cdot\frac{\beta(\alpha,\alpha_s)}{\alpha^2}\Big|_{\mathrm{NSVZ}}. \qquad (4)$$

According to [20] (see also [21] for the Abelian case), the NSVZ-like equations are valid for both renormalization group functions defined in terms of the bare couplings if the theory is regularized by higher covariant derivatives independently. In the renormalized case, this implies that these functions are defined in the HD + MSL scheme. Taking into account that the bare couplings do not depend on the renormalization scale μ, we arrive at the following exact renormalization group invariant equation

$$\Big(\alpha_{s0}\Big)^{C_2}\exp\Big(\frac{2\pi}{\alpha_{s0}} - \frac{T(R)}{\dim R}\cdot\frac{2\pi}{\alpha_0}\Big) = \Big(\alpha_s\Big)^{C_2}\times \qquad (5)$$
$$\times\exp\Big(\frac{2\pi}{\alpha_s} - \frac{T(R)}{\dim R}\cdot\frac{2\pi}{\alpha} + 3C_2\ln\frac{\Lambda}{\mu} - 2\pi C\Big) = \mathrm{RGI},$$

where for the HD + MSL scheme the parameter introduced in [22] is fixed as $C = 0$. Here RGI stands for the renormalization group invariant, i.e. the expression that vanishes after differentiaton with respect to $\ln\mu$ (at fixed values of Λ and bare couplings).

This equation is valid only for certain renormalization prescriptions which within the ways of studies in [12] and [13], form the *class* and *subgroup* of the NSVZ-type renormalization presriprtions. The latter subgroup includes the HD + MSL finite renormalization group transformation, rather than the $\overline{\mathrm{DR}}$-scheme, that violates the validity of the NSVZ-type SUSY relations at the renormalized level.

The running of the electromagnetic coupling constant upon taking into account strong interactions effects is described by the Adler D-function [4] which is related to the β-function of the coupling constant α in the limit $\alpha\to 0$ as

$$D(\alpha_s) \equiv -\frac{3\pi}{2}\frac{d}{d\ln\mu}\Big(\frac{1}{\alpha}\Big)\Big|_{\alpha_0,\alpha_{s0}=\mathrm{const},\ \alpha\to 0} = = \frac{3\pi}{2}\lim_{\alpha\to 0}\frac{\beta(\alpha,\alpha_s)}{\alpha^2}. \qquad (6)$$

The exact NSVZ-like expression for the Adler D-function in the theory under consideration was derived in [14, 15], considered in more detail in [23], and is written as

$$D(\alpha_s)\Big|_{\mathrm{NSVZ}} = \frac{3}{2}\dim R\sum_{\mathrm{a}=1}^{N_f} q_{\mathrm{a}}^2\Big(1-\gamma(\alpha_s)\Big) \equiv \equiv \frac{3}{2}\,\boldsymbol{q^2}\dim R\Big(1-\gamma(\alpha_s)\Big), \qquad (7)$$

where $\boldsymbol{q^2} \equiv \sum_{\mathrm{a}=1}^{N_f} q_{\mathrm{a}}^2$. Comparing it further with Eq. (2), we conclude that the NSVZ β-function of $\mathcal{N}=1$ SQCD can be expressed in terms of the Adler D-function by the following all-loop equation:

$$\beta_s(\alpha_s)\Big|_{\mathrm{NSVZ}} = \lim_{\alpha\to 0}\beta_s(\alpha_s,\alpha)\Big|_{\mathrm{NSVZ}} = = -\frac{\alpha_s^2}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - \frac{4\,T(R)N_f D(\alpha_s)}{3\,\boldsymbol{q^2}\dim R}\Big]. \qquad (8)$$

This equation is valid for the bare unrenormalized coupling constant $\alpha_s = \alpha_{s0}$ and for the renormalized one α_s within the NSVZ-type renormalization schemes. A similar feature will probably demonstrate itself in the supersymmetry related lattice studies of [24] and the HD + MSL like renormalization procedure, applied there but not specified in detail.

Finally, let us mention that although we do not expect the existence of exact equations relating the running of the gauge couplings in usual QCD, some features of supersymmetric theories may nevertheless hold even in the non-supersymmetric case.

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