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Topological origin of horizon temperature via the Chern–Gauss–Bonnet theorem

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This paper establishes a connection between the Hawking temperature of spacetime horizons and global topological invariants, specifically the Euler characteristic of Wick-rotated Euclidean spacetimes. This is demonstrated for both de Sitter and Schwarzschild, where the compactification of the near-horizon geometry allows for a direct application of the Chern–Gauss–Bonnet theorem. For de Sitter, a simple argument connects the Gibbon–Hawking temperature of the Bunch–Davies state to the global thermal de Sitter temperature. This establishes that spacetime thermodynamics are a consequence of the geometrical structure of spacetime itself, therefore suggesting a deep connection between global topology and semi-classical analysis.

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This paper establishes a novel connection between i) the thermal temperature of spacetime horizons and ii) the global topology of the analytically continued spacetime. Specifically, the Hawking temperature is generated by the Euler characteristic – a fundamental topological invariant – of the spacetime geometry post Wick rotation, via application of the Chern–Gauss–Bonnet theorem. This is demonstrated for the de Sitter and Schwarzschild spacetimes, verifying that both black hole and cosmological horizons possess this topological-thermodynamic relation. In the case of de Sitter, a further argument derived from this theorem clarifies that the factor-of-two difference between the Gibbons–Hawking temperature [1, 2] and the thermal process temperature emphasized by Volovik [3, 4] is of a topological origin.

The possible physical origins of the thermal properties of horizons has generated much attention (see, e.g., [5, 6]). The Gibbons–Hawking technique offers a perspective that emphasizes the role of geometry [2], together with the established fundamental relationship between path integrals and thermal field theory]1, 6, 7]. Given a spacetime containing a horizon, one performs a Wick rotation $t \to i\tau$. The original spacetime (for both de Sitter and Schwarzschild) becomes purely spatial ($-dt^2 \to +d\tau^2$). The resulting Euclidean space shares similar geometric features; namely, the horizon is still present in the appropriate limit. For example, Euclidean de Sitter in the static patch reads as

$$ds^2 = \left(1 - \frac{r^2}{l^2}\right)d\tau^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 + \\ + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2, \tag{1}$$

which diverges as $r \to l$, where $l = \sqrt{3/\Lambda}$ is the horizon radius. However, while in spacetime this is a coordinate singularity (signaling the causal horizon), for the Euclidean space (1) this is a genuine geometric singularity (conical defect) that must be "smoothed-out" in the near-horizon limit. The unique way to resolve this singularity requires the Euclidean time τ to become periodic with period $\beta = 2\pi l$ [8]. As is well known, the inverse of this period then encodes the Gibbons–Hawking temperature $T_H = 1/\beta$ of the spacetime horizon [9–12].

The periodicity in $\tau \sim \tau + \beta$ compactifies the Euclidean time τ into an angular coordinate, $\tau \in [0, \beta)$. Together with the fact that the radius r is *bounded* by the cosmological horizon, $r \in [0, l)$, this means that the near-horizon geometry of (1) is compact as a topological manifold. In particular, it has the topology of the product between the disc and the 2-sphere, $D^2 \times \mathbb{S}^2$. The consequences of this are significant, as compact spaces encode topological information into integrals of curvature scalars taken over the space (the theory of characteristic classes) [13–15]. The Euler characteristic χ is one example, which is a topological *integer* corresponding to the integral over (in vacuum) the product of curvature tensors,

$$\chi = \frac{1}{32\pi^2} \int_{D^2 \times \mathbb{S}^2} \sqrt{-g} d^4x \; R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \tag{2}$$

This is the Chern–Gauss–Bonnet theorem [14, 16, 17]. The integral can be computed explicitly, and recogniz-

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ing that the temporal integral over $\tau \in [0, \beta)$ yields the inverse Gibbons–Hawking temperature, we have

$$T_H = \frac{1}{\pi l \chi}, \quad (3)$$

which is consistent since $\chi(D^2 \times \mathbb{S}^2) = 2$. Together with a similar result for Schwarzschild [18], this argument demonstrates that the thermal temperature of horizons is sourced by a topological invariant introduced through analytic continuation: spacetime thermodynamics are of a topological origin. The argument has recently been extended to multi-horizon spacetimes [19].

Volovik has emphasized the importance of a second temperature associated to de Sitter thermodynamics, which can be considered a "global" de Sitter *state* temperature of thermal processes [3, 4]. This value is *twice* that of the Gibbons–Hawking temperature, and is motivated (for instance) by particle decay in the de Sitter environment [20]. The conceptual origin of this factor-of-two difference is non-trivial but can be explained topologically using the above techniques as follows [18]. The static patch of (1) is only one possible "slicing" of the Euclidean de Sitter space. de Sitter itself is defined as the four-dimensional hyperboloid embedded in five-dimensional Minkowski spacetime:

$$-(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 = l^2. \quad (4)$$

After Wick rotation $X^0 \to i\tau$, global Euclidean de Sitter coincides with the four-sphere $\mathbb{S}^4$. However, the four-sphere can itself be built from *two* $D^2 \times \mathbb{S}^2$ patches glued along their respective horizons at $r = l$ [21]. Being compact, $\chi(\mathbb{S}^4)$ then receives *two* contributions of the form (2) from the Chern–Gauss–Bonnet theorem. However, since $\chi(\mathbb{S}^4) = 2$ [14, 21], the only possibility for consistency is that the global de Sitter state temperature T_{dS} be *twice* that of the Gibbons–Hawking temperature, $T_{\mathrm{dS}} = 2T_H$.

Overall, this approach provides a unified geometrical framework for horizon thermodynamics. There is an inherent coupling that exists between the thermal properties of correlation functions induced through Wick rotation and a global topological invariant of the resulting compact geometry. While the horizon radiates in physical spacetime, its temperature is sourced by the topology of its analytic continuation.

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Conflict of interest. The authors of this work declare that they have no conflicts of interest.

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