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Extended Tsallis–Cirto entropy for black and white holes

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In black hole thermodynamics, entropy is non-extensive. This entropy obeys the composition rule which coincides with the composition rule in the non-extensive Tsallis–Cirto $\delta = 2$ statistics. Here we extend this approach to the thermodynamics of white holes. The entropy of the white hole is negative as follows from the rate of macroscopic quantum tunneling from black hole to white hole. The white hole entropy is with the minus sign the entropy of the black hole with the same mass, $S_{\rm WH}(M) = -S_{\rm BH}(M)$. This reflects the anti-symmetry with respect to time reversal, at which the shift vector in the Arnowitt–Deser–Misner formalism changes sign. This symmetry allows one to extend the Tsallis–Cirto entropy by adding a minus sign to the Tsallis–Cirto formula applied to white hole. As a result, the composition rule remains the same, with the only difference being that instead of entropy it contains the entropy modulus. The same non-extensive composition rule is obtained for the entropy of the Reissner–Nordström black hole. This entropy is formed by the positive entropy of the outer horizon and the negative entropy of the inner horizon. The model of the black hole formed by *black hole atoms* with Planck-scale mass is also extended to include the negative entropy of white holes.

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The thermodynamics of black holes can be obtained by studying quantum tunneling processes. Quantum tunneling of particles from a black hole determines the Hawking temperature [1]. On the other hand, macroscopic quantum tunneling, which controls the processes of dividing a black hole into smaller parts, determines the black hole entropy [2]. The black hole entropy $S_{\rm BH}(M) = 4\pi G M^2$ is non-extensive: in the process of the splitting of the black hole with mass M into two smaller black holes with masses M_1 and M_2, the entropy obeys the composition rule:

$$\sqrt{S_{\rm BH}(M = M_1 + M_2)} = \sqrt{S_{\rm BH}(M_1)} + \sqrt{S_{\rm BH}(M_2)}. \tag{1}$$

The entropy decreases in the quantum tunneling process of splitting, which is equivalent to a thermodynamic fluctuation whose probability is expressed through the difference in entropy before and after the splitting.

As discussed in [3], Eq. (1) is consistent with the non-extensive Tsallis–Cirto $\delta = 2$ entropy [4, 5]:

$$S_{\delta=2} = \sum_i p_i \left(\ln \frac{1}{p_i}\right)^2 . \tag{2}$$

The white holes are the configurations with the opposite shift vector in Arnowitt–Deser–Misner formalism. Under the time reversal the shift vector $\mathbf{v}$ changes sign, $T\mathbf{v} = -\mathbf{v}$. The entropy of the white hole can be found from the calculated rate of the macroscopic quantum tunneling from the black hole with mass M to the white hole with the same mass [2]. The obtained tunneling rate is $w \propto \exp\left(-2S_{\rm BH}(M)\right)$. Considering the process of quantum tunneling as thermodynamic fluctuation, one obtains that entropy of the white hole with mass M is with minus sign the entropy of the black hole with the same mass:

$$S_{\rm WH}(M) = -S_{\rm BH}(M) = -4\pi G M^2. \tag{3}$$

This equation reflects the antisymmetry between black and white holes with respect to time reversal. Negative entropy of a white hole confirms the statement, that white hole is the extremely fine-tuned state [6, 7].

This anti-symmetry allows one to extend the Tsallis–Cirto entropy by adding a minus sign to the Tsallis–Cirto equation, when it is applied to white hole:

$$S^{\pm}_{\delta=2} = \pm \sum_i p_i \left(\ln \frac{1}{p_i}\right)^2 . \tag{4}$$

With Eq. (4), the composition rule in Eq. (1) can be extended to describe processes that also involve white holes with their negative entropy:

$$\sqrt{|S(M_1 + M_2)|} = \sqrt{|S(M_1)|} + \sqrt{|S(M_2)|}. \tag{5}$$

The time reversal of the process of formation of a black hole is equivalent to the process of relaxation of a

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white hole. The decrease in entropy of black hole in this inverse process corresponds to the increase in entropy during the relaxation of a white hole. As a white hole relaxes, its entropy increases, starting from the most negative value, in agreement with the second law of thermodynamics.

The anti-symmetry provides the negative entropy to all the white-type horizons, including the cosmological horizon of the contracting de Sitter state and the inner horizon of the charged Reissner–Nordström (RN) black hole. For the RN black hole, the modified composition rule combines the positive entropy of the outer horizon and the negative entropy of the inner horizon. This rule shows that the total entropy of an RN black hole is completely determined by its mass M and does not depend on the conserved (integer) charge Q. This is consistent with the adiabatic transformation of the RN black hole into a Schwarzschild black hole, where the parameter α (the fine structure constant) adiabatically transforms to zero for fixed M and Q. During this adiabatic transformation, the entropy does not change.

The adiabatic approach is violated for the extremal RN black hole, since the transition to extremality is not continuous [8]. At transition from subcritical to supercritical regimes, when two horizons merge and annihilate, the entropy jumps to zero. In the supercritical regimes the black hole and its antisymmetric white clone are the same physical objects. This is in contrast to the subcritical regime, where the coordinate transformation between the black and white hole metrics has singularities at the horizons. That is why the general covariance does not apply, and the subcritical black and white holes are different physical objects with opposite entropies. The singular coordinate transformation between these objects provides another way to calculate the tunneling rate from a black hole to a white hole [2] and thus the white hole entropy.

The discontinuous transition can be considered as a type of topological quantum phase transitions, in which topological invariants change discontinuously [9]. In our case, horizons are formed, destroyed, annihilated, or change their topology [10, 11]. The corresponding topological invariant can be provided by Euler characteristic [12]. The horizon can be considered as the spacetime analogue of the Fermi surface, which undergoes Lifshitz transitions [13], while singularity is the analogue of the Weyl point in the spectrum of fermionic quasiparticles. The Weyl point can be isolated, corresponding to a naked singularity, or enclosed by the Fermi surface [14, 15].

In the same manner the time reversal transforms the expanding de Sitter with the positive Hubble parameter, $H > 0$, to contracting de Sitter with $H < 0$. Correspondingly, due to anti-symmetry of the Hubble parameter, $TH = -H$, the entropy of the volume inside the cosmological horizon is negative in the contracting Universe, $S(H < 0) = -S(H > 0) = -A/4G$, where A is the area of the cosmological horizon [16].

In all the cases considered in the paper, negative entropy occurs in the presence of a horizon: this is the horizon of a white hole, the horizon in a contracting de Sitter, and the inner horizon of a black hole. A natural question arises: can negative entropy take place in thermodynamic systems without horizons?

The process of macroscopic quantum tunneling, in which a black hole transforms into a white hole, is a thermodynamic fluctuation in which the entropy decreases to its most negative value at a fixed mass M. The further processes of the splitting of the white hole are always accompanied by increase of entropy from its most negative value. The increase of entropy is the natural process in thermodynamics, which obeys the second law of thermodynamics. This shows that it is the negative entropy which is responsible for the second law of thermodynamics for the white hole. Together with anti-symmetry, the second law of thermodynamics of a white hole justifies the emergence of negative entropy.

The Tsallis–Cirto $\delta = 2$ statistics inspires the toy model, in which the mass spectrum of the black hole is equidistant, $M = Nm_{\rm P}$, where N is an integer number and $m_{\rm P} = 1/\sqrt{8\pi G}$ is the reduce Planck mass. In this model, the black hole can be considered as ensemble of N black hole *atoms* with Planck mass scale [17]. In this toy model, the black hole *atom* has zero entropy. Thus according to the anti-symmetry in Eq. (3), the white hole *atom* also must have zero entropy. This is why we can identify a *white hole atom* with a *black hole atom*, assuming that these quanta are indistinguishable. These neutral elementary quanta serve as the building blocks of both black holes with positive entropy and white holes with negative entropy:

$$S_{\rm BH}(N) = \frac{N(N-1)}{2} = -S_{\rm WH}(N). \tag{6}$$

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1. M. K. Parikh and F. Wilczek, Phys. Rev. Lett. **85**, 5042 (2000).
2. G. E. Volovik, JETP **135**, 388 (2022).
3. G. E. Volovik, JETP Lett. **121**, 243 (2025).
4. C. Tsallis and L. J. L. Cirto, Eur. Phys. J. C **73**, 2487 (2013).

5. C. Tsallis, Entropy **22**, 17 (2020).
6. E. Witten, arXiv:2412.16795.
7. D. M. Eardley, Phys. Rev. Lett. **33**, 442 (1974).
8. S. M. Carroll, M. C. Johnson, and L. Randal, JHEP **11**, 109 (2009).
9. G. E. Volovik, Springer Lecture Notes in Physics **718**, 31 (2007).
10. T. Jacobson and S. Venkataramani, Classical Quantum Gravity **12**, 1055 (1995).
11. L. Smolin and A. Starodubtsev, arXiv:hep-th/0311163.
12. S. Liberati and G. Pollifrone, Phys. Rev. D **56**, 6458 (1997).
13. I. Lifshitz, JETP **11**, 1130 (1960).
14. G. E. Volovik and K. Zhang, J. Low Temp. Phys. **189**, 276 (2017).
15. G. E. Volovik, Phys.-Uspekhi **61**, 89 (2018).
16. G. E. Volovik, arXiv:2410.04392.
17. G. E. Volovik, arXiv:2506.13145.