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# Stability of travelling waves with constant vorticity under a strong horizontal electric field

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We study two-dimensional waves on a deep, highly permittive dielectric liquid with constant vorticity under a strong horizontal electric field, neglecting gravity and capillarity. Without vorticity, the problem is degenerate: arbitrary profiles travel without distortion at a field-determined speed and are neutrally stable. Vorticity removes this degeneracy by separating the harmonic speeds, yielding a genuine branch of travelling waves. We compute these waves from a Babenko-type equation and their spectra using the Fourier–Floquet–Hill method. Our computations show that all waves with vorticity are modulationally unstable, even at arbitrarily small amplitude, with growth rates proportional to the square of the steepness. Each branch approaches an overhanging wave whose trough collapses into a narrow neck. For co-propagating currents, a second, sharper instability arises from a four-wave resonance involving the carrier, a long counter-propagating and a short co-propagating wave.

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For a dielectric liquid of large permittivity, $\epsilon \gg 1$, in a strong tangential electric field the field lines are tangent to the free surface and the electrostatic pressure is fixed by the geometry of the interface alone. In this regime Zubarev [1] showed that, when gravity and capillarity are negligible, strongly nonlinear waves of *arbitrary* shape propagate without distortion at constant speed, and Zubarev and Zubareva [2, 3] showed that in the short-wave limit the small perturbations propagating against such a wave have real frequencies whatever the wave geometry, those propagating with it being trivially neutral, since they merely deform the profile into another exact solution. The addition to a constant vorticity permits steady waves with overhanging profiles that have no counterpart in irrotational flow [4]. It was first combined with the horizontal field in [5, 6]. What remains open is the fate of the exact Zubarev family when vorticity is switched on. In the strong-field limit, without gravity and capillarity, and in infinite depth, we show that for $\Omega \neq 0$, $\Omega$ being the shear rate of the background current, a genuine family of travelling waves exists and determine its spectral stability by the Fourier–Floquet–Hill method.

We consider a two-dimensional, inviscid, incompressible dielectric fluid of density $\rho$ and permittivity $\epsilon \gg 1$ occupying $y < \eta(x,t)$ in a horizontal electric field $E$. The background flow is a linear shear $\mathbf{U}_0 = (\Omega y, 0)$, so that $\Omega > 0$ is a current directed against a right-going wave, and the perturbation velocity is $\nabla\phi$. For $\epsilon \gg 1$ the interior electrostatic problem decouples, $\partial_n V \approx 0$ on the interface. In units with $\epsilon_0(\epsilon-1)/\rho = 1$ the governing equations are

$$\nabla^2\phi = \nabla^2 V = 0 \text{ in } y < \eta, \quad \phi \to 0 \quad (y \to -\infty), \tag{1}$$

and, on the free surface $y = \eta(x,t)$,

$$\eta_t + (\phi_x + \Omega\eta)\eta_x = \phi_y, \quad \partial_n V = 0, \tag{2}$$

$$\phi_t + \tfrac{1}{2}|\nabla\phi|^2 + \Omega\eta\phi_x - \Omega\psi = \tfrac{1}{2}\big(|\nabla V|^2 - E^2\big), \tag{3}$$

where $\psi$ is the harmonic conjugate of $\phi$. The problem is treated in conformal variables as in [5]. For $2\pi$-periodic travelling waves of speed $c$ the kinematic condition (2) then integrates exactly and the dynamic condition (3) reduces to a single scalar equation for the profile, the analogue of the Babenko equation for Stokes waves, with the Bernoulli constant known a priori. For $\Omega = 0$ this equation forces $c = E$ and leaves the profile arbitrary, recovering Zubarev's result [1]. For $\Omega \neq 0$ the linear frequencies of a wave are $\omega_{\mathrm{co,ct}}(k) = -\Omega/2 \pm \sqrt{E^2k^2 + \Omega^2/4}$, no longer homogeneous of degree one in $k$. The harmonics of a nonlinear wave travel at different speeds and the profile equation becomes a genuine nonlinear eigenvalue problem.

The $2\pi$-periodic waves are computed by Newton's method for the Fourier coefficients of the profile, $c$ and

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the Bernoulli constant, continued in the steepness. A counter-propagating current ($\Omega > 0$) deepens the trough and slows the wave, whose speed then decreases with amplitude, while $\Omega < 0$ accelerates it (Fig. 1a). Every branch terminates when the profile overhangs, at $s \approx 0.20, 0.25, 0.285$ for $\Omega = +1, +0.5, +0.25$ and $s \approx 0.365, 0.41, 0.52$ for $\Omega = -0.25, -0.5, -1$. The overhang is localised at the trough, which collapses into a narrow neck [6], and occurs here without gravity.

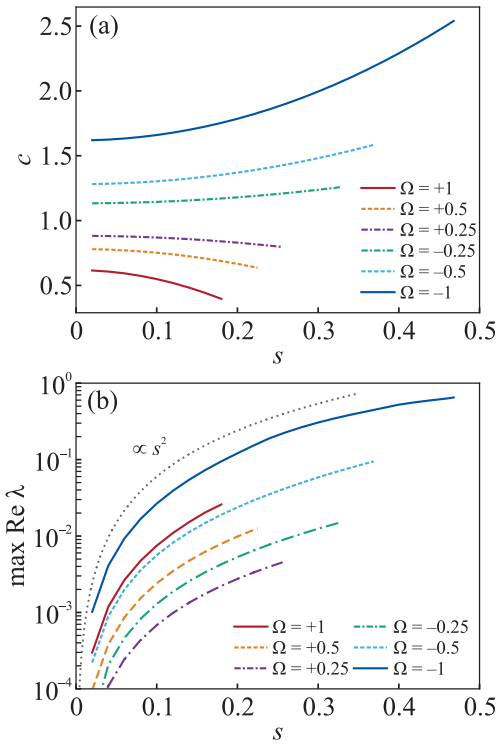



Fig. 1. (Color online) (a) – Wave speed and (b) – maximal growth rate along the travelling-wave branches for $\Omega = +1, +0.5, +0.25, -0.25, -0.5, -1$. The growth rate is the supremum over the Floquet exponent

Equations (1)–(3) are then linearised about each travelling wave in the moving frame, and the spectrum is computed by the Fourier–Floquet–Hill method. The perturbation is written as $e^{i\mu\xi}$ times a $2\pi$-periodic function, $\mu \in [-\frac{1}{2}, \frac{1}{2}]$ being the Floquet exponent, which gives a matrix eigenvalue problem for each $\mu$. For $\Omega = 0$ the moving-frame frequencies of the co-propagating branch vanish identically, and the computed spectra lie on the imaginary axis to within $10^{-11}$–$10^{-13}$ at every amplitude, confirming the short-wave result of [2, 3] nonperturbatively.

For any $\Omega \neq 0$ a small figure-eight detaches from the origin of the spectral plane already at very small amplitude and grows along the branch. The maximal growth rate (Fig. 1b) is non-zero at every amplitude, scales as $s^2$ and vanishes as $\Omega \to 0$. At fixed steepness it increases with $|\Omega|$ and is larger for a co-propagating current. The figure-eight touches the origin, the signature of a modulational (Benjamin–Feir) instability, with the classical scalings: the band edge is $O(s)$ and the maximal growth rate $O(s^2)$. The distinctive feature here is the origin of the instability. At $\Omega = 0$ the problem is exactly non-dispersive, and the weak dispersion introduced by the vorticity is what brings the sidebands into resonance with the carrier, so that every wave with $\Omega \neq 0$ is modulationally unstable, however small its amplitude.

For $\Omega < 0$ a second family of instabilities appears, with $\mathrm{Im}\,\lambda = O(1)$, as sharp peaks on top of the modulational band. In a Hamiltonian system an instability can only be born where two moving-frame frequencies $\nu(q) = qc_0 - \omega(q)$ coincide [7]. For $\Omega < 0$ this happens for the counter-propagating wave of wavenumber $\mu$ and the co-propagating wave of wavenumber $2 + \mu$, at the position of the computed peaks. The condition $\omega_{\mathrm{co}}(2 + \mu) - \omega_{\mathrm{ct}}(\mu) = 2c_0$ is a four-wave resonance of a long counter-propagating and a short co-propagating wave with two carrier waves, so the growth rate again scales as $s^2$. For $\Omega > 0$ the two branches never cross and this family is absent.

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**Conflict of interest.** The author of this work declares that he has no conflicts of interest.

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