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# UNIVERSALITY IN QUANTUM CHAOTIC SPECTRA

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We study the response of the energy levels of a quantum chaotic system to some arbitrary external perturbation. We argue that the statistical properties of the energy dispersion depends only on the mean-level spacing and a generalised conductance. A new rescaling is introduced after which the statistical correlations of the energy levels become universal. Evidence is provided from both analytical and numerical calculations.

The Wigner–Dyson distribution [1] has shown considerable success in describing the level correlations of a variety of complex systems ranging from systems with many degrees of freedom and strong interactions (such as atomic nuclei) to the quantum mechanical motion of particles in irregular potentials (such as disordered metallic grains or quantum dots). Equally, it is capable of describing Hamiltonians governed by simple dynamics such as hydrogen in an external magnetic field. In this sense the distribution is a manifestation if not a definition of quantum chaos. However, frequently we are interested in the response of the energy levels of a system to the action of some external perturbation [2–4]. Here, we introduce a simple rescaling which reveals a higher level of universality in spectral correlations.

Suppose a Hamiltonian, $\mathcal{H}$ depends on an external perturbation through some parameter $X$, having eigenvalues given by the random functions, $E_i(X)$. With no loss of generality we assume that $\langle \partial E_i(X)/\partial X\rangle = 0$, where $\langle \cdots \rangle$ denotes an average over $X$, and over a typical range of levels. We demonstrate that the rescaling,

$$x = \sqrt{C(0)}X, \qquad \epsilon_i(x) = E_i(X)/\Delta \tag{1}$$

where $\Delta$ is the mean-level spacing, and

$$C(0) = \left\langle \left( \frac{\partial \epsilon_i(X)}{\partial X} \right)^2 \right\rangle, \tag{2}$$

makes the statistics of $\epsilon_i(x)$ universal, dependent only on the Dyson ensemble. $C(0)$ describes the sensitivity of the spectrum to variations in $X$ and provides, in addition to $\Delta$, the only characteristic of the system.

The energy dissipation rate $\partial\mathcal{E}/\partial t$ caused by a time-dependent perturbation $X(t)$,

$$\frac{\partial \mathcal{E}}{\partial t} = \frac{\beta\pi}{2}\hbar C(0)\left(\frac{\partial X}{\partial t}\right)^2, \tag{3}$$

where $\beta = 1(2)$ denotes the Dyson orthogonal (unitary) ensemble, gives to $C(0)$ the physical meaning of "conductance." An analogous formula was proposed by Wilkinson [2] by making reasonable assumptions within random matrix theory. In fact, for disordered systems (3) can be derived exactly [5]. Although (3) has the form of a fluctuation-dissipation theorem we note that $C(0)$ represents mesoscopic (sample to sample, or Fermi level to Fermi level) rather than thermal fluctuations. The relation (3) can therefore be described as a mesoscopic fluctuation-dissipation theorem.

The universality can be illustrated by examining the chaotic motion of a particle scattering from either a disordered array of impurities (weakly disordered metals), or from an irregular boundary (billiard). In the former, averaging over realizations of disorder enables the rescaling to be demonstrated rigorously, and the details are presented elsewhere [5]. Results of numerical simulation presented below show that ensemble averaging is not crucial. Averaging over a typical range of energy is sufficient to ensure universality.

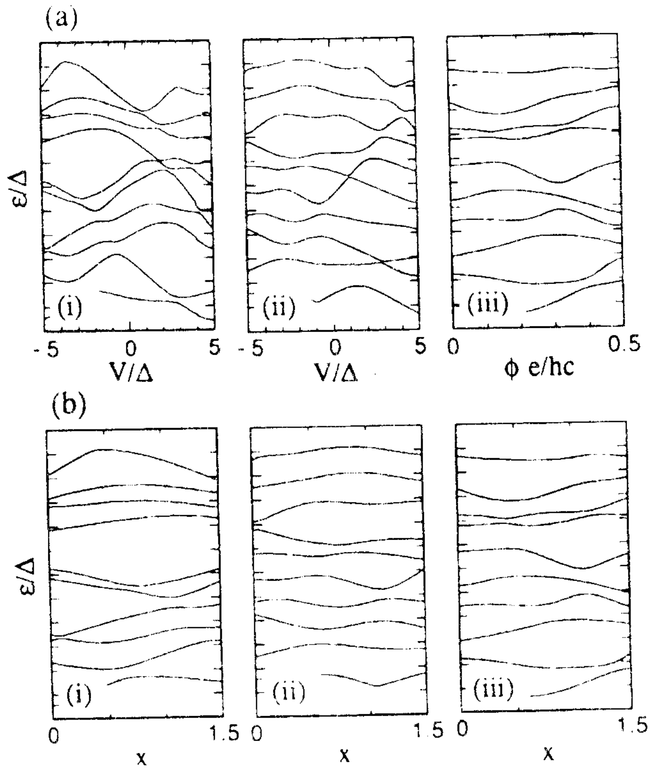


Fig.1. *(a)* Bare and *(b)* rescaled spectra shown for a typical range of energy levels for *(i)* $X \equiv V/E_c$, *(ii)* $X \equiv V/E_c$, and *(iii)* $X \equiv \phi e/hc$. In *(i)* and *(ii)* the scattering is from impurities with $W = 2.4$, while in *(iii)* it is from by an irregular boundary with geometry shown inset in Fig. 2. Case (ii) differs from *(i)* in that an applied magnetic field breaks $T$-invariance making the symmetry of *(ii)* unitary. After rescaling the unitary samples, *(ii)* and *(iii)* become statistically equivalent, and distinct from the orthogonal sample, *(i)*

Firstly, we can apply an external perturbation in the form of an Aharonov–Bohm flux through a ring. As particles circulate around the ring the wavefunction acquires a phase, $2\pi\phi e/hc \equiv 2\pi X$. Substitution of (2) in (3) gives the Thouless formula [6] for conductance $G = e^2C(0)/2h$ [5] which has the form recently proposed in Ref.[7].

The change of sign of $\phi$ under $T$-reversal implies unitary symmetry. We will also examine a second external perturbation which can act on a system taken from either an orthogonal or unitary ensemble. Applying a potential step across a sample (with half the sites raised and half lowered by a potential $V$) and setting $X \equiv V/E_c$ we obtain [5] $G = 12\pi^2 e^2 C(0)/h$. A fixed magnetic field can be used to drive the system from an orthogonal to a unitary ensemble (reducing $C(0)$ by a factor of 2). According to universality, after rescaling with (1) the spectra should become statistically indistinguishable from other unitary ensembles. Comparison with $X \equiv \phi e/hc$ therefore provides a critical test of the universality.

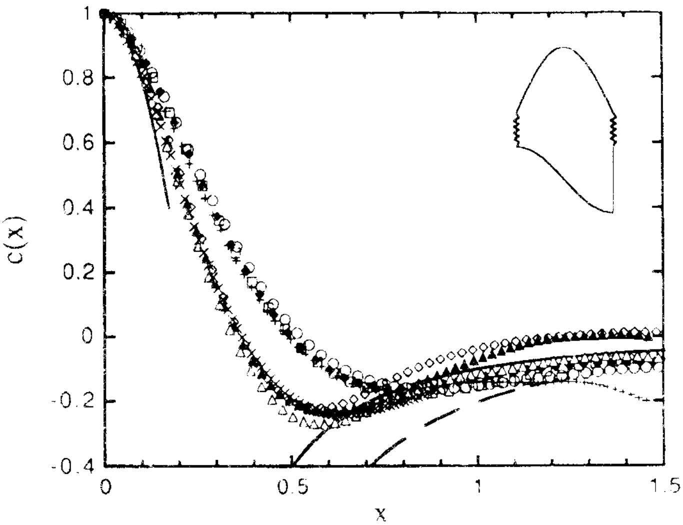


Fig.2 $c(x)$ measured from disordered and chaotic samples for a variety of cases. Measurements with $X \equiv V/E_c$ are shown for a range of disorder; $W = 1.9$ (open circles), $W = 2.4$ (filled diamonds), and $W = 2.9$ (open squares) at zero magnetic field; $W = 2.4$ (open diamonds) at non-zero field; and for a chaotic billiard at zero field (crosses +), and at non-zero field (crosses ×). Measurements with $X \equiv \phi e/hc$ are shown for disorder $W = 2.4$ (filled triangles), and for a chaotic billiard (open triangles). The chaotic billiard (shown inset) is assumed to be connected along the zigzag edge. The asymptotic approximations to $c(x)$ are shown for the unitary (continuous) and orthogonal (broken) ensembles. All measurements from samples with impurity scattering are averaged over four realizations of the disorder

The simulations were performed with a tight-binding Anderson model with on-site energies chosen randomly from the range $-W/2 < W_i < W/2$ ($W = 0$ for the billiard). The universality is illustrated qualitatively in Fig. 1 where the rescaling is applied to three different spectra. A quantitative test of the rescaling is provided by the autocorrelation function of level "velocities,"

$$c(x) = \left\langle \frac{\partial \epsilon_i(\bar{x} + x)}{\partial \bar{x}} \frac{\partial \epsilon_i(\bar{x})}{\partial \bar{x}} \right\rangle, \tag{4}$$

shown in Fig. 2. All the data from both the billiard and disordered samples collapse onto one of two curves according to the Dyson ensemble. In particular, $c(x)$ measured with $X \equiv V/E_c$ at zero magnetic field collapses onto the "orthogonal curve," while at non-zero field it follows the "unitary curve," coinciding with the flux autocorrelator ($X \equiv \phi e/hc$).

We have not succeeded in evaluating $c(x)$ analytically, and the behavior of the function is available only in asymptotic region of large $x$, where [4,8,5] $c(x) =$

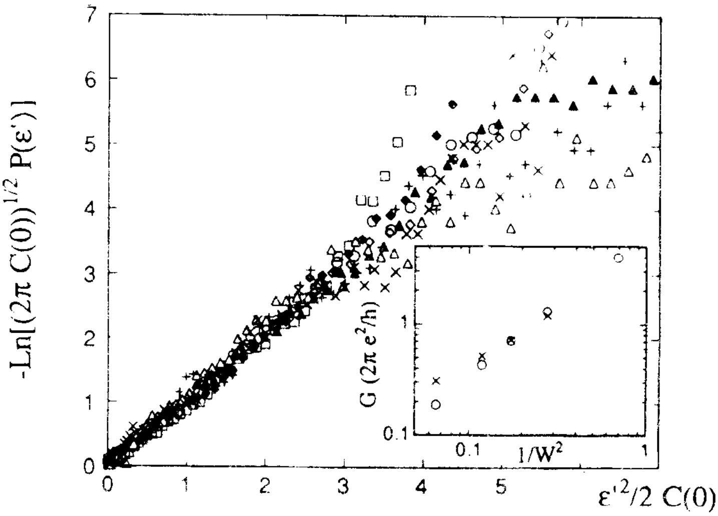


Fig.3. Measurements of the velocity distribution function $P(\epsilon')$, where $\epsilon' \equiv \partial\epsilon/\partial x$ with symbols corresponding to that used in Fig. 2. The variation of conductance, $G$ as a function of disorder $1/W^2$ is shown inset with $X \equiv \phi e/hc$ (open circles), and $X \equiv V/E_c$ (crosses $X$). All measurements for impurity scattering are averaged over four realizations of the disorder

$-2/\beta\pi^2x^2$, and for the unitary ensemble, $c(x) = 1 - 2\pi^2x^2 + \mathcal{O}(x^4)$ at vanishing $x$. These results can be obtained from the autocorrelator of density of states fluctuations,

$$k(\omega, x) = \left\langle (1/\mathcal{V}^2) \sum_{ij} \delta(\epsilon - \epsilon_i(\bar{x}))\ \delta(\epsilon - \omega - \epsilon_j(x + \bar{x})) \right\rangle - 1$$

known exactly for both orthogonal and unitary ensembles [5]. For example, in the unitary ensemble,

$$k_u(\omega, x) = \mathrm{Re} \int_{\pi}^{\infty} d\lambda_1 \int_{-\pi}^{\pi} d\lambda \frac{1}{2\pi^2} \exp\left[x^2(\lambda^2 - \lambda_1^2) + i\omega(\lambda - \lambda_1)\right]. \qquad (5)$$

The expression $k(\omega, x)$ can be used to show that the distribution of level velocities, $P(\partial\epsilon/\partial x)$ is Gaussian with unit variance for both unitary and orthogonal ensembles. This prediction is confirmed by simulation (Fig. 3) over a wide range of velocities. The variation of the conductance as a function of disorder is shown inset, and is consistent with the approximate $1/W^2$ dependence of $G$ predicted by the Born approximation [9].

In conclusion we have argued that the dispersion of spectra of quantum chaotic systems in response to an arbitrary external perturbation depends on two parameters, the mean-level spacing, $\Delta$ and a generalised conductance, $C(0)$. Moreover, we have proposed and verified a rescaling in which the dependence on these parameters can be removed. We suggest that the universality applies to all classes of chaotic system with the same generality as the Wigner–Dyson distribution.

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