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BLACK HOLE SOLUTION IN 2D GRAVITY WITH TORSION

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The $2D$ model of gravity with zweibeins e^a and the Lorentz connection one-form $\omega^a{}_b$ as independent gravitational variables is considered and it is shown that the classical equations of motion are exactly integrated in coordinate system determined by components of $2D$ torsion. For some choice of integrating constant the solution is of the charged black hole type. The conserved charge and ADM mass of the black hole are calculated.

Recently much attention has been payed to the investigation of two-dimensional dilaton gravity. This is mainly inspired by string theory, and also by the fact that it gives the simplest model for the dynamical description of a two-dimensional gravity [1-5], the gravitational variables are dilaton and metric fields $(\phi, g_{\mu\nu})$. In empty (without matter) space the classical equations of motion are exactly integrated [1-3] and the solution describes the two-dimensional black hole. On the quantum level it was shown [4] that this model is renormalizable. One can consider the $2D$ dilaton gravity as "toy model" for the study of old problems of black hole formation and evaporation [5].

On the other hand, numerous recent attempts to formulate the theory of gravity in the framework of a consistent gauge approach resulted in constructing the gauge gravity models for the de Sitter and Poincaré groups (for a review see, e.g., [6]). The independent variables are now vielbeins $e^a = e^a_\mu dx^\mu$ and Lorentz connection one-form $\omega^a{}_b = \omega^a{}_{b,\mu} dx^\mu$. The application of these methods in two dimensions was justified by attempts to give an alternative description of two-dimensional dynamical gravity in terms of variables $(e^a, \omega^a{}_b)$. It was argued also that investigation of simple two-dimensional model leads to a better understanding of four-dimensional gravity and its quantization [7]. It was shown in [7] that the Lagrangian $L = \gamma R^2 + \beta T^2 + \lambda$ is the most general one quadratic in curvature R

and torsion T, and containing a cosmological constant λ. The classical equations of motion were analyzed in conformal gauge [7] and in light cone gauge [8] and their exact integrability was demonstrated.

In this note we will consider the model for two dimensional de Sitter gravity. The constants γ, β, λ are fixed in this case with only one free parameter α^2 and the action is of the Yang-Mills type [6].

1. In two dimensions the gauge gravity is described in terms of zweibeins $e^a = e^a_\mu dz^\mu, a = 0, 1$ (the $2D$ metric on the surface M^2 has the form $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$) and Lorentz connection one-form $\omega^a{}_b = \omega\varepsilon^a{}_b$, $\omega = \omega_\mu dz^\mu$ ($\varepsilon_{ab} = -\varepsilon_{ba}$, $\varepsilon_{01} = 1$). The de Sitter curvature two-form $\mathcal{R}$ [6] in two dimensions takes the form:

$$\mathcal{R} = \begin{pmatrix} R\epsilon^a{}_b + \alpha^2 e^a \wedge e_b & \alpha T^a \\ \alpha T_b & 0 \end{pmatrix}$$

where α is the coupling constant, and curvature and torsion two-forms are: $R = d\omega$, $T^a = de^a + \varepsilon^a{}_b \omega \wedge e^b$.

The dynamics of gravitational variables (e^a, ω) is determined by the action of the Yang-Mills type [6]:

$$S = \int_{M^2} \frac{1}{4} Tr * \mathcal{R} \wedge \mathcal{R} =$$

$$= \int_{M^2} \frac{\alpha^2}{2} * T^a \wedge T^a + \frac{1}{2} * R \wedge R - \frac{\alpha^4}{4} \varepsilon_{ab} e^a e^b + \alpha^2 R \qquad (1)$$

where $*$ is the Hodge dualization. The last term in (1) is the boundary one and it does not affect the equations of motion.

Let us consider variables $\rho = *R$ and $q^a = *T^a$. Variation of action (1) with respect to zweibeins e^a and Lorentz connection ω leads to the following equations of motion:

$$d\rho = -\alpha^2 q^a \varepsilon_{ab} e^b \qquad (2)$$

$$\nabla q^a = -\frac{1}{2\alpha^2} [\rho^2 + \alpha^2 q^2 - \alpha^4] \varepsilon^a{}_b e^b, \qquad (3)$$

where $\nabla q^a \equiv dq^a + \omega \varepsilon^a{}_b q^b$.

2. One particular solution of (2)-(3) is evident. Assuming $q^2 = constant$ one gets from (2)-(3), provided e^a are linearly independent everywhere on M^2: $\rho^2 = \alpha^4$, $q^a = 0$ in all points of the two-dimensional manifold. That is, torsion is zero and M^2 is the de Sitter space.

Let now q^2 be nonconstant, and hence non zero identically everywhere in M^2. Then from eqs.(2)-(3) we have the following equation connecting q^2 and ρ:

$$\frac{dq^2}{d\rho} = \frac{1}{\alpha^4} \Phi, \qquad (4)$$

where $\Phi(\rho, q^2) = \rho^2 + \alpha^2 q^2 - \alpha^4$.

The solution of this equation has the form:

$$q^2(\rho) = -\frac{1}{\alpha^2}(\rho + \alpha^2)^2 + \epsilon e^{\frac{\rho}{\alpha^2}}, \qquad (5)$$

where ϵ is integrating constant, we will see that it is proportional to ADM mass. Notice that due to pseudoeuclidean signature q^2 can take both positive and negative values.

One can see that for large negative ρ independently of the value of integration constant ϵ function $q^2(\rho)$ has the asymptotics $q^2 \sim -\frac{1}{\alpha^2}(\rho+\alpha^2)^2$. The form of this function for positive ρ depends on the constant ϵ.

A. $\epsilon > 0$.

In this case for large positive ρ function q^2 is positive and approximately $q^2 \sim \epsilon e^{\frac{\rho}{\alpha^2}}$.

The critical points of function $q^2(\rho)$ (5) (where $\frac{dq^2}{d\rho} = 0$) are solutions of following equation:

$$\rho_c = -\alpha^2 + \frac{\epsilon}{2} e^{\frac{\rho_c}{\alpha^2}}. \tag{6}$$

One can show that there are no such points for $\epsilon > 2\alpha^2$; for $\epsilon = 2\alpha^2$ one gets one critical point $\rho_c = 0$; for $0 < \epsilon < 2\alpha^2$ the function has two critical points: the first one is positive ($\rho_{c1} > 0$) and the second is negative ($\rho_{c2} < 0$).

In general case $q^2(\rho)$ in critical point is equal to the following value:

$$q_c^2 = \frac{\epsilon}{2} e^{\frac{\rho_c}{\alpha^2}} (1 - \frac{\rho_c}{\alpha^2}) \tag{7}$$

One can see that q_c^2 is positive if $\rho_c < 0$ (since $\epsilon > 0$). The sign of q^2 in positive critical point ρ_{c1} depends on value of constant ϵ. If ϵ is slightly smaller than $2\alpha^2$ then q_{c1}^2 is still positive. The point ρ_{c1} is a minimum which goes down with decreasing constant ϵ and reaches zero value $q_{c1}^2 = 0$ if, as follows from (7), $\rho_{c1} = \alpha^2$. One can see from (6) that it corresponds to [1] $\epsilon = \frac{4\alpha^2}{e}$. Thus we come to following conclusion about the behavior of function $q^2(\rho)$.

For $\epsilon > \frac{4\alpha^2}{e}$ the function $q^2(\rho)$ has only one zero at a negative $\rho < -\alpha^2$. If $\epsilon = \frac{4\alpha^2}{e}$ there are two such zeros: at $\rho < -\alpha^2$ and $\rho = \alpha^2 > 0$. For $0 < \epsilon < \frac{4\alpha^2}{e}$ the function $q^2(\rho)$ vanishes at three points: one for $\rho < -\alpha^2$ and two for $\rho > -\alpha^2$ (one of which satisfies $\rho > \alpha^2$).

B. $\epsilon = 0$.

In this case function (5) reduces to $q^2 = -\frac{1}{\alpha^2}(\rho + \alpha^2)^2$ which is negative everywhere except for a point $\rho = -\alpha^2$ where it vanishes.

C. $\epsilon < 0$.

As one can see from (5) the function $q^2(\rho)$ has no zeros in this case and it is negative for any ρ. Evidently there is only one critical point (maximum) ρ_c which lies in the interval $-\alpha^2 - \frac{|\epsilon|}{e} < \rho_c < -\alpha^2$.

3. Thus eqs.(2)-(3) determine q $(= \sqrt{q^2})$ as a function of ρ. Further analysis of (2) easily shows that $\xi(q) = 0$, where we denoted one-form $\xi = q_c e^c$. Let us use this and introduce a new coordinate system which is (pseudo)polar with q playing the role of a 'radial' coordinate, while the 'angular' coordinate ϕ is then clearly such that its differential is proportional to ξ. Assuming (for definiteness) that $q^2 = (q^0)^2 - (q^1)^2 > 0$, one can write the torsion components in the form: $q^0 = q\cosh\phi$, $q^1 = q\sinh\phi$.

Let us consider q, ϕ as the new local coordinates on M^2. The differentials $\{dq, d\phi\}$ form basis in the space of one-forms. Since q is a function of ρ, we can

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[1] $e = 2.7\ldots$ is the Euler number

use an equivalent basis $\{d\rho,\ d\phi\}$. From the construction of q, ϕ (see above) and (2)-(3) we get

$$q^a \varepsilon_{ab} e^b = -\frac{d\rho}{\alpha^2},$$
$$q_a e^a = \xi = B d\phi, \tag{8}$$

where B is some function of variables ρ and ϕ.

Solving equations (2),(3) we find finally

$$B = q^2 B_0 \exp\left(-\frac{\rho}{\alpha^2}\right),$$

where B_0 is an arbitrary function of ϕ. Consequently the metric has the form:

$$d^2 s = (e^0)^2 - (e^1)^2 = q^2(\rho) \exp\left(-\frac{2\rho}{\alpha^2}\right)(d\phi)^2 - \frac{1}{\alpha^4 q^2(\rho)}(d\rho)^2, \tag{9}$$

where $q^2(\rho)$ is known function (5), and without any loose of generality we redefined the 'angular' variable $B_0(\phi)d\phi \rightarrow d\phi$.

4. Let us remind that when varying the action in order to get an equation of motion one usually drops out the surface term which arises when integrating by parts. The correct way of doing so is to impose appropriate boundary conditions. Assuming the variations $\delta\omega$ and δe^a are arbitrary at spatial infinity (which in the polar coordinate system corresponds to the infinite value of the usual radial coordinate) one gets from the action (1) the boundary conditions:

$$\rho|_\infty = -\alpha^2; \quad q^a|_\infty = 0. \tag{10}$$

The constraint that torsion at space infinity is zero is too strong. It leads to the constraint $\epsilon = 0$ in (5), so most of solutions are omitted.

Let us add to the action (1) following term:

$$S_b = -a\alpha^2 \int_{\partial M^2} \gamma^{1/2} d\tau \tag{11}$$

where $\gamma = \det\gamma_{\mu\nu}$, $\gamma_{\mu\nu} = g_{\mu\nu} - \kappa n_\mu n_\nu$ is metric induced on the boundary ∂M^2 with normal vector n_μ $(n_\mu n_\nu g^{\mu\nu} = \kappa)$, $\kappa = 1$ for spacelike boundary and $\kappa = -1$ for timelike boundary. Then variation of total action $S_{tot} = S + S_b$ (note that it is still positive in euclidean signature for $a > 0$) leads to the modified boundary conditions at space infinity, which for metric (9) take the form

$$\rho|_\infty = -\alpha^2; \quad q|_\infty = a. \tag{12}$$

It means that integrating constant in (5) $\epsilon = a^2 e$. Thus the found solution describes the two-dimensional asymptotically de Sitter space with two kinds of possible singularities: where $\rho = -\infty$ and $\rho = +\infty$.

5. The most interesting solution is of the type A with ρ laying in the interval $-\alpha^2 \leq \rho < +\infty$. One can see that metric (9) describes the two-dimensional asymptotically de Sitter space-time with singularity ($\rho = +\infty$) and horizons at points where the function $q^2(\rho)$ has zeros.

As was described above, for $\epsilon > \frac{4\alpha^2}{e}$ such points are absent and we have naked singularity. For $0 < \epsilon < \frac{4\alpha^2}{e}$ we obtain two horizons which coincides when $\epsilon = \frac{4\alpha^2}{e}$.

Thus the metric (9) for $0 < \epsilon \leq \frac{4\alpha^2}{e}$ resembles the charged two-dimensional black hole type solution [3] (though the (9) is not exactly metric considered in [3] for $2D$ dilaton gravity coupled with Maxwell field) . The case $\epsilon = \frac{4\alpha^2}{e}$ corresponds to the extremal black hole.

In support of this analogy we note that the equation (2) is similar to the Maxwell equation $df = *j$ where $f = *(dA)$ is strength of abelian gauge field A and j is charged matter current one-form. Then the second gravitational equation (3) is similar to the equation of motion for charged matter. It is not surprising because the local Lorentz symmetry in two dimensions is abelian and analogous to the $U(1)$-symmetry of Maxwell theory.

From eq.(3) we get that the corresponding Lorentz current one-form $*J = -\alpha^2 q^a \varepsilon_{ab} e^b$ is conserved, $d * J = 0$. Integrating $*J$ over any spacelike hypersurface Σ we get that total charge $Q = \int_\Sigma *J$ is equal to curvature ρ at infinity[2)]:

$$Q = \rho|_\infty \tag{13}$$

and consequently for the boundary conditions (12) the total charge $Q = -\alpha^2$.

6. To calculate the ADM mass for the black hole solution (9) let us assume [3] that only the equation for ω (2) is satisfied and consider the zweibein energy-momentum one-form $T^a = T^a_\mu dz^\mu$ which can be determined as follows: $\delta_e S = \int - * T^a \wedge \delta e^a$. For action (1) it takes the form:

$$\tilde{T}^a \equiv - * T^a = \alpha^2 \nabla q^a + \tfrac{1}{2}[\rho^2 + \alpha^2 q^2 - \alpha^4]\varepsilon^a{}_b e^b \ .$$

Multiplying this expression on $q^a \exp(-\frac{\rho}{\alpha^2})$ we obtain that

$$\begin{aligned} T = \tilde{T}^a q^a \exp(-\frac{\rho}{\alpha^2}) = \\ = \alpha^2 \exp(-\frac{\rho}{\alpha^2})(\frac{1}{2}dq^2 - \frac{1}{2\alpha^4}(\rho^2 + \alpha^2 q^2 - \alpha^4)d\rho) \end{aligned} \tag{14}$$

is obviously conserved: $dT = 0$. It implies that there exist such a scalar function m that

$$T = dm. \tag{15}$$

Straightforward calculations show that the mass function m at point ρ can be written in the following explicit form:

$$m = \frac{\alpha^2}{2} \exp(-\frac{\rho}{\alpha^2})(q^2 + \frac{1}{\alpha^2}(\rho + \alpha^2)^2). \tag{16}$$

In the case when the field equations $*T^a = 0$ are satisfied eq.(17) implies that $m = constant$ and for $q^2(\rho)$ in the form (5) we get that $m = \frac{\alpha^2 \epsilon}{2}$.

Hence only the solution of the type A describes the positive mass configuration (solutions of the type B and C have correspondingly zero and negative mass).

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2)Note that the same formula is valid in (1+1)-electrodynamics: $Q \equiv \int_\Sigma *j = f|_\infty$ [3]

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